Inductive limits of locally m-convex algebras

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Abstract

In this short note we prove that the locally convex inductive limit of commutative locally m-convex algebras is already locally m-convex whenever multiplication is continuous.

A topological algebra is an (associative) algebra endowed with a vector space topology such that multiplication is jointly continuous. If one considers inductive limits (in the category of topological vector spaces) of topological algebras multiplication may fail to be continuous since it is a *bilinear* map on the product which is certainly not linear, and forming inductive limits only respects the linear structure (although this is quite clear, the difference between linearity and bilinearity in this context has been overlooked several times in the literature).

The most important category of topological algebras is that of locally m-convex (l.m.c.) algebras having a 0-neighbourhood basis consisting of m-convex sets (i.e. absolutely convex sets B which are multiplicative in the sense that $B^2 \subseteq B$). This class was introduced by Michael [9] and Arens [2] since it allows generalizations and applications of the theory of Banach algebras.

Since their work a number of authors studied the question whether the inductive limit (in the category of locally convex spaces) of l.m.c. algebras is again l.m.c., see e.g. [1, 3, 4, 5, 7, 8, 10]. This is indeed true for countable inductive limits of normed algebras [1, 4]. On the other hand, Warner [10] gave an example of an inductive limit of metrizable l.m.c. algebras which is not l.m.c., and in [4] it is shown that there are strict inductive limits of l.m.c. Fréchet algebras having even discontinuous multiplication.

At several places in the literature it is claimed that the inductive limit of l.m.c. algebras is a topological algebra, and then the problem is considered when it is again l.m.c.

Received by the editors November 2002.

Communicated by F. Bastin.

2000 *Mathematics Subject Classification :* 46J05, 46M40.

Key words and phrases : locally m-convex algebras, inductive limits.

We now show that the situation is exactly opposite!

If the (countable locally convex) inductive limit of commutative l.m.c. algebras is a topological algebra then it is already l.m.c.

The only fact about countable locally convex inductive limits we need here is that the sets $\Gamma(\bigcup_{n\in\mathbb{N}} U_n)$ where U_n are 0-neighbourhoods of the steps (and $\Gamma(B)$ denotes the absolutely convex hull of the set B) form a basis of the 0-neighbourhood filter of the inductive limit. More information can be found e.g. in Köthe's book [6].

Convexity and the multiplicative structure fit together well as the following simple lemma shows (see e.g. [9, lemma 1.3]).

Lemma. *Let B, C be two subsets of an algebra. Then $\Gamma(B)\Gamma(C) \subseteq \Gamma(BC)$. In particular, the absolutely convex hull of a multiplicative set is m-convex.*

The proof follows easily from the fact that $\Gamma(B)$ consists of all absolute convex combinations $\sum \lambda_i b_i$ with $b_i \in B$ and scalars λ_i such that $\sum |\lambda_i| \leq 1$.

Theorem. *Let $A = \mathrm{ind}_n A_n$ be the inductive limit of a sequence of commutative l.m.c. algebras such that multiplication is continuous on A. Then A is locally m-convex.*

Proof. Let U be an absolutely convex 0-neighbourhood in A. We will construct an m-convex 0-neighbourhood W which is contained in U. Using the continuity of multiplication we find inductively 0-neighbourhoods V_k with $V_{k+1}V_{k+1} \subseteq V_k \subseteq U$ for $k \in \mathbb{N}_0$. Since the steps A_n are l.m.c. there are m-convex 0-neighbourhoods $V_{n,k}$ in A_n such that for all $n, m \in \mathbb{N}$ and $k \in \mathbb{N}_0$ we have

$$V_{n,k+1} \subseteq V_{n,k} \subseteq V_k \text{ and } V_{n,k+1}V_{m,k+1} \subseteq \Gamma(\bigcup_{l\in\mathbb{N}} V_{l,k}).$$

Proceeding by induction on $r \in \mathbb{N}$ we will show for $1 \leq k(1) < k(2) < \ldots < k(r)$ and $n(1), \ldots, n(r+1) \in \mathbb{N}$

$$(\star) \qquad V_{n(1),k(1)}V_{n(2),k(2)} \cdots V_{n(r),k(r)}V_{n(r+1),k(r)} \subseteq U.$$

For $r = 1$ this is certainly true, and if $(\star)$ holds for $r-1$ we obtain

$$\begin{aligned}
& V_{n(1),k(1)} \cdots V_{n(r),k(r)}V_{n(r+1),k(r)} \\
\subseteq \;& V_{n(1),k(1)} \cdots V_{n(r-1),k(r-1)}\Gamma(\bigcup_{l\in\mathbb{N}} V_{l,k(r)-1}) \\
\subseteq \;& V_{n(1),k(1)} \cdots V_{n(r-1),k(r-1)}\Gamma(\bigcup_{l\in\mathbb{N}} V_{l,k(r-1)}) \\
\subseteq \;& \Gamma(\bigcup_{l\in\mathbb{N}} V_{n(1),k(1)} \cdots V_{n(r-1),k(r-1)}V_{l,k(r-1)}) \subseteq U
\end{aligned}$$

by the induction hypothesis.

We now define $W_n := V_{n,n}$. Since $W_{k(r)} \subseteq V_{k(r),k(r-1)}$ $(\star)$ yields

$$W_{k(1)}W_{k(2)} \cdots W_{k(r)} \subseteq U$$

for all $1 \leq k(1) < k(2) < \ldots < k(r)$. If $m(1), \ldots, m(s)$ are arbitrary natural numbers we choose $1 \leq k(1) < k(2) < \ldots < k(r)$ such that $\{m(1), \ldots, m(s)\} = \{k(1), \ldots, k(r)\}$. Using the commutativity of A and the m-convexity of W_n we obtain

$$W_{m(1)} \cdots W_{m(s)} = \prod_{i=1}^{r} \prod_{m(j)=k(i)} W_{m(j)} = \prod_{i=1}^{r} W_{k(i)}^{|\{j:\, m(j)=k(i)\}|} \subseteq \prod_{i=1}^{r} W_{k(i)} \subseteq U.$$

This yields for every $s \in \mathbb{N}$

$$\left(\bigcup_{m \in \mathbb{N}} W_m \right)^s = \bigcup_{(m(1),\ldots,m(s)) \in \mathbb{N}^s} W_{m(1)} \cdots W_{m(s)} \subseteq U \text{ and therefore}$$

$$W := \Gamma \left(\bigcup_{s \in \mathbb{N}} \left(\bigcup_{m \in \mathbb{N}} W_m \right)^s \right) \subseteq U.$$

Since by the lemma the absolutely convex hull of multiplicative sets is m-convex we have found an m-convex 0-neighbourhood W which is contained in U. Therefore, A is locally m-convex. ∎

To finish this note we would like to mention that we neither know whether commutativity is essential for our theorem nor what happens for general non-countable inductive limits.

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