On generalized topological function algebra bundles

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Abstract

We consider function algebras whose elements are functions taking values from topological algebra bundles and study section algebras and their spectra of the topological algebra bundles involved. This naturally extends the so far standard theory of vector-valued function algebras to an appropriate *varying set-up.*

Introduction

Topological algebra bundles have been considered in [3,4]. By this we mean a triple $\xi = (E, \pi, X)$, with $\pi : E \longrightarrow X$ a continuous map between the corresponding topological spaces, whose fibre is a topological algebra, while the appropriate local triviality conditions are also in force. Our aim is to transfer standard results of vector-valued functions to the framework of topological algebra bundles (cf. §5), this being achieved as an application of tensor product topological algebra bundles. Yet, our main technique is to consider vector-valued function spaces, in terms of topological tensor products (cf. for instance, [6 : Chapter XI]), always within topological algebra bundle theory. Thus, for appropriate topological algebra bundles, we examine the analysis of the section algebra of a tensor product algebra bundle, as the $\mathcal{C}_c(X)$-tensor product of the section algebras of the factor bundles (Theorem 3.1).

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On the other hand, the *spectrum bundle* of a given topological algebra bundle (Definition 4.1), is realized as the *spectrum* (Gel'fand space) of the algebra of sections of the given algebra bundle. Thus, the spectrum (bundle) of a tensor product algebra bundle, appears as the "Whitney sum" of the factor spectrum bundles (Theorem 4.2). As applications we consider in the last section the tensor product algebra bundle of a topological algebra (the latter viewed as a constant topological algebra bundle) and a topological algebra bundle, in general (cf. (5.1)), and further examine the form of the sections and that of the corresponding spectrum bundle at issue (Theorems 5.1, 5.2). In this connection, we consider the sections and spectra of the bundle of continuous (holomorphic, $\mathcal{C}^\infty$-) ξ-valued functions on X, for a given algebra bundle ξ over X (cf. 5.(1), 5.(2), 5.(3)), as well as, the *group algebra bundle* (cf. 5.(4)) of a locally compact group G, relative to ξ.

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1 Preliminaries

In [3: § 1] we have given the notion of a *topological algebra bundle* over a topological space X. This is a triple $\xi = (E, \pi, X)$ with $\pi : E \to X$ continuous, whose fibre $\xi_x \equiv \pi^{-1}(x)$ is a topological ($\mathbb{C}$-)algebra, whereas the standard "local triviality" conditions are satisfied [3: Definition 1.1]. More generally, one can replace the (structure) group $Aut(M)$ of $\mathbb{C}$-automorphisms of the fibre $M = \pi^{-1}(x), x \in X$, of ξ with a topological group acting continuously and effectively on $M(: G < Aut(M))$. We denote such a bundle, by

$$\xi = (E, \pi, X, M, G) \ . \tag{1.1}$$

Locally $(m-)$convex, unital, commutative, nuclear, algebra bundles etc. will have the obvious meaning, referring to the pertinent class of topological algebras for the fibres.

Now, by a *system of transition functions* of a topological space X we mean a set

$$\Lambda \equiv \{(U_i), (h_{ij}) \ , \ M, \ G\}_{i \in J} \tag{1.2}$$

consisted of an open covering (U_i) of X, a topological algebra M and a G-cocycle (h_{ij}) , associated to (U_i), with G a topological group as above.

Thus, a characterization of topological algebra bundles, in terms of a system of transition functions, can be given, as was also the case in [3 : Theorem 1.1] (see also [8 :Theorem 1.1]). Namely, let $\Phi_G(X)$ be the set of isomorphism classes of topological algebra bundles over X (cf. (1.1)), having locally equicontinuous structure group, say G, and let $H^1(X, G)$ be the set of equivalence classes of systems of transition functions on X (cf.(1.2)). Then, one has

$$\Phi_G(X) = H^1(X, G) \tag{1.3}$$

within a bijection (ibid.).

2 Tensor product topological algebra bundles

In this section we consider tensor product algebra bundles, providing also a characterization for it (Theorem 2.2). First some useful preliminary remarks are necessary.

Let G, H be topological groups acting continuously and effectively on the unital locally convex algebras M, N respectively. If $l_G : G \times M \longrightarrow M,\ \ l_H : H \times N \longrightarrow N$ are the corresponding (continuous) actions, the map

$$l_{G\times H} : (G \times H) \times (M \underset{\pi}{\otimes} N) \longrightarrow M \underset{\pi}{\otimes} N : \Big((g,h), \sum_{i=1}^{n} x_i \otimes y_i\Big) \tag{2.1}$$

$$\longmapsto \sum_{i=1}^{n} gx_i \otimes hy_i$$

is an algebra representation of the group $G \times H$ on $M \underset{\pi}{\otimes} N$ (:projective tensor product algebra, [6: Chapter X, Lemma 3.1]), such that one gets the next result.

Lemma 2.1. *Let G, H be topological groups acting continuously and effectively on the unital locally convex algebras M, N respectively. Then, the group $G \times H$ acts continuously and effectively on $M \underset{\pi}{\otimes} N$.*

Proof. let $g \in G$ with $g \neq e_G$ (neutral element of G). Then there exists $x \in M$ with $gx \neq x$, such that $mx \otimes e_H \neq x \otimes e_H$ for any $m \in G \times H$ (e_H the neutral element of H), with $m \neq (e_G, e_H)$. Moreover, if $\phi : M \times N \longrightarrow M \underset{\pi}{\otimes} N$ is the canonical continuous map of the relative tensor product, one has

$$\Big(\phi \circ (l_G \times l_H)\Big)\Big((g,h),(x,y)\Big) = \phi(gx, hy) = gx \otimes hy =$$

$$l_{G\times H}\Big((g,h),(x \otimes y)\Big) = \Big(l_{G\times H} \circ (id_{G\times H} \circ \phi)\Big)\Big((g,h),(x,y)\Big)\ ,$$

such that the linearity of the relative maps led us to

$$\phi \circ (l_G \times l_H) = l_{G\times H} \circ (id_{G\times H} \circ \phi) \tag{2.2}$$

The corresponding topologies on $G \times H$ (cartesian product) and $M \underset{\pi}{\otimes} N$ (:projective tensor product) give the continuity of (2.1). ∎

Now, let G be a topological group acting continuously and effectively on a topological algebra M having continuous multiplication and consider $Aut(\hat{M})$ the group of automorphism of $\hat{M}$ (:completion of M) endowed with the relative topology from $Hom_s(\hat{M}, \hat{M})$ (: continuous endomorphisms of $\hat{M}$ with the simple convergence topology on $\hat{M}$, cf. [6: Chapter V, 8(1)]). Under suitable conditions (e.g., if G is locally compact), G acts continuously on $\hat{M}$. Moreover, if G is locally equicontinuous and $\hat{G}$ is the subgroup of $Aut(\hat{M})$ consisting of $g \in G$, extended to the completion of M, then $\hat{G}$ is also locally equicontinuous (cf. [6: Chapter V, Theorem 8.1]), such that $\hat{G}$ acts continuously and effectively on $\hat{M}$. In this regard, consider $M \hat{\underset{\pi}{\otimes}} N$ the complete locally convex algebra, completion of the projective tensor product algebra of two given unital locally convex algebras with continuous multiplications M, N. From [6: Chapter XIII, (8.44)] and subsequent remarks therein, if any automorphism of

$M\hat{\underset{\pi}{\otimes}}N$ leaves either one of the factor algebras M, N invariant one has the following decomposition

$$Aut(M\hat{\underset{\pi}{\otimes}}N) = Aut(M) \times Aut(N). \tag{2.3}$$

As an immediate result of the last relation and the above comments (cf. also [6: Chapter V, Theorem 8.1]) one has the next.

Proposition 2.1. *Let M, N be two unital locally convex algebras with continuous multiplications and $\hat{M}, \hat{N}$ their corresponding completions. Moreover, consider G, H two locally compact groups acting continuously and effectively on M, N respectively. Then $G\hat{\times}H$ is locally equicontinuous subgroup of $Aut(M\hat{\underset{\pi}{\otimes}}N)$ if and only if this is the case for G, H.* ■

In the sequel we will define the tensor product of two locally convex algebra bundles and study some characteristic properties of it.

So, let $\xi = (E, \pi, X, M, G)$ and $\eta = (F, \pi, X, N, H)$ be two unital locally convex algebra bundles with continuous multiplications and locally equicontinuous structure groups. Moreover, let $\Lambda_\xi = \{(U_i), (\lambda_{ij}), M, G\}$, $\Lambda_\eta = \{(U_i), (\mu_{ij}), N, H\}$ be the corresponding transition systems, such that for any pair (i, j) we consider the map

$$\lambda_{ij} \otimes \mu_{ij} : U_i \cap U_j \longrightarrow G \times H : x \mapsto (\lambda_{ij} \otimes \mu_{ij})(x) := \lambda_{ij}(x) \otimes \mu_{i_j}(x) \ . \tag{2.4}$$

This is a continuous map, since continuity of λ_{ij}, μ_{ij} implies that one of the relative tensor product map, while the next conditions are also satisfied:

$$(\lambda_{ii} \otimes \mu_{ii})(x) = id_{M\otimes N} \ , \text{ for each } \ x \in U_i \tag{2.5}$$

$$(\lambda_{ij} \otimes \mu_{ij})(x) \circ (\lambda_{jk} \otimes \mu_{jk})(x) = (\lambda_{ik} \otimes \mu_{ik})(x), \text{ for each } \ x \in U_i \cap U_j \cap U_k \ . \tag{2.6}$$

The last assertion is an immediate consequence of the very definitions (cf. also [3:(1.10), (1.11)] and [6: Chapter X, § 3]. Thus, considering the corresponding completions, one gets the map

$$\lambda_{ij}\hat{\otimes}\mu_{ij} : U_i \cap U_j \longrightarrow G\hat{\times}H\Big(< Aut(M\hat{\underset{\pi}{\otimes}}N)\Big) \tag{2.7}$$

which is continuous and moreover satisfies the relative conditions (2.5), (2.6). So $(\lambda_{ij}\hat{\otimes}\mu_{ij})_{(i,j)}$ defines a $G\hat{\times}H$*-cocycle on* $(U_i)_i$, such that one gets the corresponding bundle (cf. (1.3)) having bundle space $E\hat{\otimes}F \equiv \dot{\bigcup_i}\Big(U_i \times M\hat{\underset{\pi}{\otimes}}N\Big)/\sim$ (cf. [3: Theorem 1.2]), which is also isomorphic to the direct sum $\sum_x\Big(\pi^{-1}(x)\hat{\otimes}\rho^{-1}(x)\Big)$, projection map (denoted by) $\pi\hat{\otimes}\rho : E\hat{\otimes}F \longrightarrow X$, base space X, fibre $M\hat{\underset{\pi}{\otimes}}N$ and group $G\hat{\times}H$.

Definition 2.1.. Let $\xi = (E, \pi, X, M, G)$, $\eta = (F, p, X, N, H)$ be two unital locally convex algebra bundles with continuous multiplications and locally equicontinuous structure groups. The bundle

$$\xi\hat{\otimes}\eta \equiv (E\hat{\otimes}F, \ \pi\hat{\otimes}p, X, M\hat{\underset{\pi}{\otimes}}N, G\hat{\times}H) \tag{2.8}$$

as above, is the *tensor product topological algebra bundle of* ξ, η.

Now, we give a *converse-like statement* of the above. That is, suppose we have two unital locally convex algebras with continuous multiplications M, N, a topological space X and an open covering (U_i) of X. Moreover, let K be a topological group, acting continuously and effectively on $M \hat{\underset{\pi}{\otimes}} N(: K < Aut(M \hat{\underset{\pi}{\otimes}} N)\big)$ and $(\tilde{h}_{ij})$ a K-cocycle of (U_i). The question which arises is the following:

(2.9) *Under the above hypotheses, can we define two locally convex algebra bundles having* M, N *as fibres ?*

The *answer is in the positive*: The map

$$\mu : M \longrightarrow M \hat{\underset{\pi}{\otimes}} N : m \longmapsto \mu(m) := m \otimes 1_N \tag{2.10}$$

(1_N is the unit of N), is a continuous one-to-one map, such that M is embedded in $M \hat{\underset{\pi}{\otimes}} N$ $(: M \overset{\mu}{\hookrightarrow} M \hat{\underset{\pi}{\otimes}} N)$. Thus, for any $x \in U_i \cap U_j$, we consider the automorphism $\tilde{h}_{ij}(x) \in K(< Aut(M \hat{\underset{\pi}{\otimes}} N))$ and define

$$h_{ij}(x) := \tilde{h}_{ij}(x)|_M \ . \tag{2.11}$$

The map (2.11) is, of course, an automorphism of M, i.e, $h_{ij}(x) \in Aut(M)$, while we also set, by definition

$$G := \{h_{ij}(x), \quad h_{ij}(x) = \tilde{h}_{ij}(x)|_M, \quad h_{ij}(x) \in K, \quad x \in U_i \cap U_j\} \tag{2.12}$$

The set (2.12) is a topological group (subgroup of $Aut(M)$) acting continuously and effectively on M. Moreover, the relation (2.11) and the hypotheses for $(\tilde{h}_{ij})$ prove that (h_{ij}) satisfy the relations

$$h_{ii}(x) = id_M, \quad x \in U_i \tag{2.13}$$

$$h_{ij}(x) \circ h_{jk}(x) = h_{ik}(x), \quad x \in U_i \cap U_j \cap U_k \ . \tag{2.14}$$

Moreover, the continuity of

$$h_{ij} : U_i \cap U_j \longrightarrow G : x \longmapsto h_{ij}(x) := \tilde{h}_{ij}(x)|_M \tag{2.15}$$

follows from the continuity of the maps

$$U_i \cap U_j \xrightarrow{\tilde{h}_{ij}} K \quad \xrightarrow{p} G \tag{2.16}$$

since (2.15) is expressed as the composition of p (: canonical projection map $Aut(M \hat{\underset{\pi}{\otimes}} N) \to Aut(M)$, cf. (2.3), restricted in K) and $\tilde{h}_{ij}$ as in (2.11). Thus, one gets the following set of transition functions

$$\Lambda \equiv \{(U_i), (h_{ij}), M, G\}_{i \in J} \tag{2.17}$$

(cf. (2.11), (2.13), (2.15)).

Similarly, we can also define a H-cocycle

$$l_{ij} : U_i \cap U_j \longrightarrow H(< Aut(N)) : x \longmapsto l_{ij}(x) := \tilde{h}_{ij}(x)|_N \tag{2.18}$$

with $\quad H := \{l_{ij}(x) : l_{ij}(x) := \tilde{h}_{ij}(x)|_N, \quad \tilde{h}_{ij}(x) \in K, \quad x \in U_i \cap U_j\}$

a topological group, acting continuously and effectively on N (cf (2.12)), such that the following system of transition functions

$$\Lambda' \equiv \{(U_i), (l_{ij}), N, H\}_{i \in J} \tag{2.19}$$

is well defined. Now, for any $x \in U_i \cap U_j$, one gets the following decomposition

$$\tilde{h}_{ij}(x) = h_{ij}(x) \otimes l_{ij}(x), \quad x \in U_i \cap U_j \tag{2.20}$$

under suitable conditions (cf. for instance [6: Chapter XIII, (8.24)] and also (2.3)).

Theorem 2.1. *Let M, N be unital locally convex algebras with continuous multiplications and (U_i) an open covering of a topological space X. Moreover, let K be a locally equicontinuous topological group, acting continuously and effectively on $M \hat{\underset{\pi}{\otimes}} N$ and $(\tilde{h}_{ij})$ a K- cocycle on (U_i), such that (2.3) be valid. Then, there exist two locally convex algebra bundles on X of fibre type M, N, respectively, whose tensor product is exactly the bundle defined by the given K-cocycle.*

Proof. The hypotheses imply the existence of a system of transition functions

$$\tilde{\Lambda} \equiv \{(U_i), (\tilde{h}_{ij}),\ M \hat{\underset{\pi}{\otimes}} N,\ K\} \tag{2.21}$$

from which one defines the transition systems (2.17), (2.19).
Thus, from (1.3) one gets the corresponding locally convex algebra bundles $\xi = (E, \pi, X, M, G), \eta = (F, p, X, N, H)$, with $E \equiv \dot{\bigcup_i}(U_i \times M)/\sim$, $\pi : E \longrightarrow X$(canonical projection), $F \equiv \dot{\bigcup_i}(U_i \times N)/\sim, p : F \to X$. Moreover, let $\tilde{\xi}$ be the locally convex algebra bundle defined through (2.21). Then, (2.20) gives

$$\tilde{\xi} = \xi \hat{\otimes} \eta \tag{2.22}$$

(cf. Definition 2.1), such that the assertion is proven . ∎

An immediate consequence of the above Theorem (cf. (2.21) and also (2.20)) is given by the following.

Corollary 2.1 . *Let M, N be unital locally convex algebras with continuous multiplications and K a locally equicontinuous topological group acting continuously and effectively on $M \hat{\underset{\pi}{\otimes}} N$ such that (2.3) be valid. Then, each topological algebra bundle on X of fibre type $M \hat{\underset{\pi}{\otimes}} N$ is expressed as a tensor product of locally convex algebra bundles on X having M, N as fibres, respectively.* ∎

Theorem 2.1 and Corollary 2.1 give us the next *characterization of tensor product algebra bundles in terms of tensor product algebras.*

Theorem 2.2. *Let M, N be unital locally convex algebras with continuous multiplications, such that (2.3) is valid, and let X be a topological space. Then, the two following assertions are equivalent*:

1) *There exist locally convex algebra bundles on X of fibre type M, N, respectively, with locally equicontinuous structure groups.*

2) *There exists a locally convex algebra bundle on X of fibre type $M \hat{\underset{\pi}{\otimes}} N$ and locally equicontinuous structure group.*

■

The previous result can still be expressed in the following succinct form (I am indebted to A. Mallios for this):

> *A locally convex tensor product algebra bundle* is *"decomposable"*, in the sense of (2.3), if and only if this happens for the respective *"structure group"* of the bundle at issue.

Scholium 2.1. Concerning the above results, we can still make analogous considerations using more "algebraic hypotheses", in the sense that the topological space X can be replaced by the spectrum (Gel'fand space [6:Chapter V,Definition 1.1]) $\mathcal{M}(\mathbb{A})$ of a locally convex algebra $\mathbb{A}$ or the spectrum of one of the given algebras M, N. Thus, we can take a form of a fibre tensor product bundle as in [5: § 1],

$$\lambda = (\mathbb{A}, M, N, K) \tag{2.23}$$

with K a topological group, acting continuously and effectively on $M \hat{\underset{\pi}{\otimes}} N$, satisfying the next two conditions:
First,

(2.24) *There exists an arbitrary family (I_i) of closed 2-sided ideals of $\mathbb{A}$ with compact hulls $(h(I_i))$ and non-empty interiors, say $U_i \equiv (h(I_i))^\circ$, such that $\mathcal{M}(\mathbb{A}) = \dot{\bigcup_i} U_i$ (: topological sum)*

and second,

(2.25) *There exists a K-cocycle associated to $\mathcal{U} \equiv (\cup_i)_i$*, viz. *an element $(\tilde{h}_{ij}) \in Z'(\mathcal{U}, K)$.*

Condition (2.25) is weaker than the corresponding one in [5: (1.2)]. In this case we can repeat the above technique, taking analogous results.

3 The algebra of sections

In this section we mainly examine the relation between the algebra of sections of $\xi \hat{\otimes} \eta$ and those of the factor bundles ξ, η as above (cf. (2.8)).

First we recall that given a locally convex algebra bundle ξ on X of fibre type M and structure group G, the corresponding locally convex algebra of sections, say $\Gamma(\xi)$ (cf. [3:(2.4)]), is isomorphic to the locally convex algebra

$$\mathcal{B} \equiv \{\tau = (\tau_i)_i : \tau_i(x) = h_{ij}(x)\tau_j(x),\ x \in U_i \cap U_j\} \tag{3.1}$$

this being a subalgebra of $\prod_i \mathcal{C}_c(U_i, M)$, with $(U_i)_i$ an open covering of X and (h_{ij}) the corresponding G-cocycle on (U_i) (cf. [4: (3.1)] and also (1.3)). That is one has

$$\Gamma(\xi) = \mathcal{B} \tag{3.2}$$

within an isomorphism of locally convex algebras (cf. [3: Theorem]). Moreover, $\Gamma(\xi)$ is a *locally convex* $\mathcal{C}_c(X)$*-algebra*, where $\mathcal{C}_c(X)$ denotes the (locally m-convex) algebra of $\mathbb{C}$-valued continuous functions on X, endowed with the compact open topology (cf. [3: p. 406]). Now, let ξ, η be locally convex algebra bundles and $\Gamma(\xi), \Gamma(\eta)$ the corresponding locally convex $\mathcal{C}_c(X)$-algebras of sections. Thus, one considers the projective $\mathcal{C}_c(X)$-tensor product algebra

$$\Gamma(\xi) \underset{\pi}{\otimes}{}_{\mathcal{C}_c(X)} \Gamma(\eta) \tag{3.3}$$

(cf. [1: Proposition 5.1]). Moreover, if the fibres of ξ, η have continuous multiplications, the algebra (3.3) has the same condition such that its completion

$$\Gamma(\xi) \underset{\pi}{\hat{\otimes}}{}_{\mathcal{C}_c(X)} \Gamma(\eta) \tag{3.4}$$

is a complete locally convex $\mathcal{C}_c(X)$-algebra (cf. [1: Proposition 5.1 (5.6)]).

Theorem 3.1. *Let ξ, η be unital commutative nuclear complete locally m-convex algebra bundles with locally equicontinuous structure groups over a locally compact base space, having an open covering of σ-compact subsets. Then,*

$$\Gamma(\xi \hat{\otimes} \eta) = \Gamma(\xi) \underset{\tau}{\hat{\otimes}}{}_{\mathcal{C}_c(X)} \Gamma(\eta) \tag{3.5}$$

($\tau \equiv \pi = \varepsilon$) *within an isomorphism of locally convex $\mathcal{C}_c(X)$-algebras.*

Proof. Let M, N be the fibres of ξ, η respectively and $(U_i)_i$ the open covering of X. For any U_i one has

$$\mathcal{C}_c(U_i, M \underset{\tau}{\hat{\otimes}} N) = \mathcal{C}_c(U_i \times_{U_i} U_i, M \underset{\tau}{\hat{\otimes}} N) = \mathcal{C}_c(U_i, M) \underset{\tau}{\hat{\otimes}}{}_{\mathcal{C}_c(U_i)} \mathcal{C}_c(U_i, N) \tag{3.6}$$

within locally m-convex $\mathcal{C}_c(U_i)$-algebras (cf. [2:Theorem], [6: p.391, (1.16)]).

Then, considering the corresponding cartesian products we have the next isomorphisms of locally m-convex $\prod\limits_i \mathcal{C}_c(U_i)$- algebras

$$\begin{aligned}\prod_i \mathcal{C}_c(U_i, M \underset{\tau}{\hat{\otimes}} N) &= \prod_i (\mathcal{C}_c(U_i, M) \underset{\tau}{\hat{\otimes}}{}_{\mathcal{C}_c(U_i)} \mathcal{C}_c(U_i, N)) \\ &= ([4 : \text{p.}60, (5.6)]) \prod_i \mathcal{C}_c(U_i, M) \underset{\tau}{\hat{\otimes}}{}_{\prod_i \mathcal{C}_c(U_i)} \prod_i \mathcal{C}_c(U_i, N)\end{aligned}$$

such that the assertion follows from (3.1), (3.2) (cf. also [6: Chapter XI, Theorem 1.1]). ∎

The isomorphism (3.5) is also an immediate consequence of [4: Theorem 3.1], setting α the complex line bundle $\mathbb{C}$, since in this case $\Gamma(\mathbb{C}) = \mathcal{C}_c(X)$.

4 The spectrum bundle

In [3: § 4] we have introduced the notion of the *spectrum bundle* of a topological algebra bundle, as a *fibre bundle* over a topological space, of fibre type the spectrum of the fibre of the given algebra bundle. More precisely, given a topological group G, acting continuously and effectively on a topological algebra M, we consider the set

$$ {}^tG := \{{}^tg : g \in G\} \subseteq Aut\Big(\mathcal{M}(M)\Big) \tag{4.1} $$

such that ${}^tg(\chi) := \chi \circ g, \quad \chi \in \mathcal{M}(M)$. The set ${}^tG$ becomes a topological group acting continuously and effectively on $\mathcal{M}(M)$, taking, for instance, M semi-simple and $\mathcal{M}(M)$ locally equicontinuous (cf. [3: (3.2)]). Thus, given a system of transition functions on X, say $\Lambda \equiv \{(U_i)_i, (\lambda_{ij}), M, G\}$, one has a system of transition functions on X

$$ \Lambda^\star \equiv \{(U_i)_i, (\lambda^\star_{ji}), \mathcal{M}(M), {}^tG\} \tag{4.2} $$

with

$$ \lambda^\star_{ji}(x) := {}^t\Big(\lambda_{ij}(x)\Big), \quad x \in U_i \cap U_j \ . \tag{4.3} $$

Now, given a topological algebra bundle $\xi = (E, \pi, X, M, G)$, we consider (h_{ij}), its corresponding G-cocycle on X and $(h^\star_{ji})$ the relative “dual” functions given by (4.3). We take the set

$$ S \equiv \dot{\bigcup_i}\Big(U_i \times \mathcal{M}(M)\Big)/\sim \ , \tag{4.4} $$

where the *equivalence relation* “$\sim$” on the disjoint union is given by

$$ (x_i, \chi_i) \sim (x_j, \chi_j) \ \text{ iff } \quad \begin{cases} x_i &= x_j \equiv x \in U_i \cap U_j \\ \chi_j &= \Big(h^\star_{ji}(x)\Big)(\chi_i) \end{cases} \tag{4.5} $$

(cf. [3: (3.4),(3.5)]). Moreover, one gets the canonical projection map $p : S \longrightarrow X$ such that we set the following.

Definition 4.1. Let $\xi = (E, \pi, X, M, G)$ be a *topological algebra bundle*, such that the set ${}^tG$ (cf. (4.1)) is a topological group acting continuously on $\mathcal{M}(M)$. The bundle, say $\mathcal{M}(\xi)$, defined through the transition functions $(h^\star_{ji})$ as above is called the *spectrum bundle of* ξ. That is

$$ \mathcal{M}(\xi) \equiv (S, p, X, \mathcal{M}(M), {}^tG) \ . \tag{4.6} $$

In order to examine the decomposition of $\mathcal{M}(\xi \hat{\otimes} \eta)$ we recall the following basic result.

Theorem 4.1.*[3: Theorem 4.1]. Let* $\xi = (E, \pi, X, M, G)$ *be a commutative complete Gel'fand-Mazur Waelbroeck algebra bundle, whose base is locally compact, having an open relatively compact covering* (U_i). *Then, one has*

$$\mathcal{M}(\xi) = \dot{\bigcup_i}\Big(U_i \times \mathcal{M}(M)\Big)/ \sim \cong \mathcal{M}\big(\Gamma(\xi)\big) \tag{4.7}$$

within a homeomorphism . ∎

The interest of the homeomorphism (4.7) lies in the "algebraization" of the notion of the spectrum bundle, in the sense that the study of the last notion is equivalent to that one of the spectrum of the corresponding algebra of sections. An application of the last comments gives the next.

Theorem 4.2. *Let* ξ, η *be two topological algebra bundles over* X, *such that the conditions of Theorem 3.1 be valid. Moreover, suppose that the spectra of* $\Gamma(\xi), \Gamma(\eta)$ *are locally equicontinuous. Then,*

$$\mathcal{M}(\xi \hat{\otimes} \eta) \cong \mathcal{M}(\xi) \oplus \mathcal{M}(\eta) \equiv \mathcal{M}(\xi) \times_X \mathcal{M}(\eta) \tag{4.8}$$

within an isomorphism of fibre bundles. (The second member of (4.8) means "Whitney sum").

Proof. Theorem 4.1 gives the homeomorphism

$$\mathcal{M}(\xi \hat{\otimes} \eta) = \mathcal{M}\Big(\Gamma(\xi \hat{\otimes} \eta)\Big) \tag{4.9}$$

while [1: Theorem 2.1] implies the next homeomorphism

$$\mathcal{M}\Big(\Gamma(\xi) \hat{\otimes}^{\tau}_{\mathcal{C}_c(X)} \Gamma(\eta)\Big) = \mathcal{M}\Big(\Gamma(\xi)\Big) \times_{\mathcal{M}(\mathcal{C}_c(X))} \mathcal{M}(\Gamma(\eta)) \ . \tag{4.10}$$

But [6: Chapter VII, Theorem 1.2]) entails the homeomorphism $\mathcal{M}(\mathcal{C}_c(X)) = X$, such that the relation (4.10) gives

$$\mathcal{M}\Big(\Gamma(\xi) \hat{\otimes}^{\tau}_{\mathcal{C}_c(X)} \Gamma(\eta)\Big) = \mathcal{M}\Big(\Gamma(\xi)\Big) \times_X \mathcal{M}(\Gamma(\eta)) \ . \tag{4.11}$$

within a homeomorphism.Thus (3.5),(4.9),(4.11) and (4.6) establish the assertion. ∎

5 Applications

In this section we consider applications of the above, concerning standard function algebras (continuous, holomorphic, differentiable etc), replacing algebra-valued with bundle-valued functions.

First, some more terminology is needed. Thus, we consider tensor product algebra bundles (Definition 2.1), with one of the factors being a *constant bundle*. That is, let $\mathbb{A}$ be a locally convex algebra with continuous multiplication and

$\xi = (E, \pi, X, M, G)$ a bundle of locally convex algebras with continuous multiplication. The algebra $\mathbb{A}$ can be considered as a locally convex algebra bundle, having bundle space $X \times \mathbb{A}$, base space X, $p : X \times \mathbb{A} \to X$ the projection map, fibre $\mathbb{A}$ and group $\{id_{\mathbb{A}}\}$, the identity map of $\mathbb{A}$. That is, we consider $\mathbb{A} \equiv (X \times \mathbb{A}, p, \mathbb{A}, \{id_{\mathbb{A}}\})$. Definition 2.1 gives the next *locally convex algebra bundle*

$$\mathbb{A}\hat{\otimes}\xi \equiv (\mathbb{A}\hat{\otimes}E, \tilde{\pi}, X, \mathbb{A}\underset{\pi}{\otimes}M, \mathbb{A}\hat{\otimes}G) \tag{5.1}$$

with $\mathbb{A}\hat{\otimes}E \equiv \sum_{x\in X} \alpha\hat{\otimes}\pi^{-1}(x) \cong \dot{\bigcup_i}(U_i \times (\mathbb{A}\underset{\pi}{\otimes}M))/\sim$, $(\alpha \in \mathbb{A})$ and $\tilde{\pi} : \mathbb{A}\hat{\otimes}E \longrightarrow X$ the canonical projection map. This is the so called *tensor product algebra bundle of* $\mathbb{A}$ *and* ξ. The bundle (5.1) is constructed through the $\mathbb{A}\hat{\otimes}G$-cocycle

$$\begin{aligned} \tilde{h}_{ij} := id_{\mathbb{A}}\hat{\otimes}h_{ij} : &U_i \cap U_j \longrightarrow \mathbb{A}\hat{\otimes}G : \\ &x \mapsto \tilde{h}_{ij}(x) := id_{\mathbb{A}}\hat{\otimes}h_{ij}(x) \end{aligned} \tag{5.2}$$

with (h_{ij}) being the G-cocycle defined via the bundle ξ (cf. (1.2)).

Theorem 3.1 and relation (5.1) (cf. also [6:Chapter XI, Theorem 1.1]) give the following

Theorem 5.1. *Let* $\mathbb{A}$ *be a unital commutative complete nuclear locally* m*-convex* Q*- algebra and* ξ *a unital commutative complete locally* m*-convex* Q*-algebra bundle over a locally compact base space, having an open covering of* σ*-compact subsets with locally equicontinuous structure group. Then,*

$$\Gamma(\mathbb{A}\hat{\otimes}\xi) = \mathbb{A}\underset{\tau}{\otimes}\Gamma(\xi) \tag{5.3}$$

$(\tau \equiv \pi = \varepsilon)$ *within an isomorphism of locally convex algebras.* ■

Moreover the spectrum of the bundle (5.1) is given from the next Theorem. The proof is an immediate consequence of (5.1), (5.3) and Theorem 4.2.

Theorem 5.2. *Under the hypotheses of Theorem 5.1 and the local equicontinuity of the spectra of* $\mathbb{A}, \Gamma(\xi)$*, one has*

$$\mathcal{M}(\mathbb{A}\hat{\otimes}\xi) = \mathcal{M}(\mathbb{A}) \times \mathcal{M}(\xi)\ , \tag{5.4}$$

within an isomorphism of fibre bundles. ■

The second member of (5.4) is the bundle space of the fibre bundle $\mathcal{M}(\xi)$ *defined on (the base space)* $\mathcal{M}(\mathbb{A})$.

In this regard, the results of §§2, 3 can also be considered for the bundle (5.1). So, under the hypotheses of Theorem 2.2, any locally convex algebra bundle $\tilde{\xi}$ on X as in (2) of Theorem 2.2 has the decomposition

$$\tilde{\xi} = \mathbb{A}\hat{\otimes}\xi\ , \tag{5.5}$$

for ξ a locally convex algebra on X, like (1) of Theorem 2.2.

Moreover, we consider a *fibre tensor product bundle*

$$\lambda = (\mathbb{A}, M, G) \tag{5.6}$$

(cf. Scholium 2.1 for $G = K, M = N \equiv M$), with $\mathbb{A}, M$ unital locally convex algebra bundles, satisfying (2.24), (2.25). Thus, one defines a locally convex algebra bundle ξ_λ on $\mathcal{M}(\mathbb{A})$, of fibre type M, having G as structure group. The relation between the last bundle and $\tilde{\xi}$, defined through Theorem 2.1, is given by the next bundle isomorphism

$$\tilde{\xi} = \mathbb{A}\hat{\otimes}\xi_\lambda \tag{5.7}$$

(cf. (5.5)).

Now, we will give the forms of some *generalized function algebra bundles*: Thus, given a suitable locally convex algebra $\mathbb{A}$, the algebra of $\mathbb{A}$-valued continuous functions on X, i.e $\mathcal{C}_c(X, \mathbb{A})$, can be expressed as a tensor product $\mathcal{C}_c(X)\underset{\varepsilon}{\hat{\otimes}}\mathbb{A}$, with $\mathcal{C}_c(X) \equiv \mathcal{C}_c(X, \mathbb{C})$ (cf. [6:Chapter XI, Theorem 1.1]). The same situation is also valid for the algebras $\mathcal{C}_c^\infty(X, \mathbb{A})$ (: differentiable functions, [6: Chapter XI, Theorem 2.1]), $\mathcal{O}(X, \mathbb{A})$ (: holomorphic functions, [6: Chapter XI, Lemma 4.1])), $L^1(G, \mathbb{A})$ (: group algebra of a locally compact group G relative to $\mathbb{A}$, cf. [6: Chapter XI, (5.14)]). So, through the above analysis and the relation (5.1) (cf. also Theorems 5.1, 5.2) we have the following results:

5.(1). *The bundle* $\mathcal{C}_c(X, \xi)$. Let X be a *locally compact space* and $\mathcal{C}_c(X) \equiv \mathcal{C}_X$ the *locally m-convex algebra*, as above. For a *locally m-convex algebra bundle* ξ one gets (cf (5.1)),

$$\mathcal{C}(X, \xi) := \mathcal{C}_X\hat{\otimes}\xi\ , \tag{5.8}$$

the so called *bundle of continuous ξ-valued functions on X*. In this regard, under the hypotheses of Theorem 5.1, the corresponding algebra of sections is given by

$$\Gamma(\mathcal{C}_c(X, \xi)) \cong \mathcal{C}_X\underset{\tau}{\hat{\otimes}}\Gamma(\xi) \cong \mathcal{C}_c(X, \Gamma(\xi)) \tag{5.9}$$

($\tau \equiv \pi = \varepsilon$), *within isomorphisms of locally m-convex algebras.* Moreover, under the hypotheses of Theorem 5.2, the *spectrum bundle of* (5.8) is given by

$$\mathcal{M}(\mathcal{C}(X, \xi)) = X \times \mathcal{M}(\xi) \tag{5.10}$$

within *a fibre bundle isomorphism* (cf. also Theorem 4.1).

5.(2). *The bundle* $\mathcal{C}^\infty(X, \xi)$. Let X be a $\mathcal{C}^\infty$- manifold and $\mathcal{C}_c^\infty(X) \equiv \mathcal{C}_X^\infty$ the locally m-convex algebra of $\mathbb{C}$-valued differentiable functions on X in the “Schwartz topology” (cf. [6: Chapter IV, 4.(2)]). For a *locally m-convex algebra bundle* ξ on X we have

$$\mathcal{C}^\infty(X, \xi) := \mathcal{C}_X^\infty\hat{\otimes}\xi\ , \tag{5.11}$$

the so called *bundle of $\mathcal{C}^\infty$- functions on X with values in ξ*. The algebra of sections of (5.11), under the hypotheses of Theorem 5.1, is given by

$$\begin{aligned}\Gamma(\mathcal{C}^\infty(X, \xi)) = \Gamma(\mathcal{C}_X^\infty\hat{\otimes}\xi) &= \mathcal{C}_X^\infty\underset{\tau}{\hat{\otimes}}\Gamma(\xi) \\ &= \mathcal{C}^\infty(X, \Gamma(\xi))\end{aligned} \tag{5.12}$$

($\tau \equiv \pi = \varepsilon$) within *isomorphisms of locally m-convex algebras* (cf. also [6: Chapter XI, Theorem 2.1] and Theorem 5.1). Moreover, under the hypotheses of Theorem 5.2, the *spectrum bundle* of (5.11) is given by

$$\mathcal{M}(\mathcal{C}^\infty(X,\xi)) = X \times \mathcal{M}(\xi) \tag{5.13}$$

within an isomorphism of fibre bundles. Indeed, Theorem 5.2 and relation (5.12) give the next homeomorphisms

$$\begin{aligned}\mathcal{M}(\mathcal{C}^\infty(X,\xi)) &= \mathcal{M}(\Gamma(\mathcal{C}^\infty(X,\xi)) = \mathcal{M}(\mathcal{C}^\infty_X \underset{\tau}{\hat{\otimes}} \Gamma(\xi))) \\ &= \mathcal{M}(\mathcal{C}^\infty(X) \times \mathcal{M}(\Gamma(\xi)) = X \times \mathcal{M}(\xi)\end{aligned} \tag{5.14}$$

5.(3). *The bundle $\mathcal{O}(X,\xi)$.* If X is a *Stein space* and $\mathcal{O}(X) \equiv \mathcal{O}_X$ the *locally m-convex algebra* of $\mathbb{C}$-valued holomorphic functions on X (cf. [6: Chapter IV, 4.(3)]), for a *locally m- convex algebra bundle* ξ, one defines

$$\mathcal{O}(X,\xi) := \mathcal{O}_X \hat{\otimes} \xi \tag{5.15}$$

the *bundle of holomorphic ξ-valued functions on X*. Then, for suitable conditions (cf. Theorem 5.1 and also [6: Chapter X, Lemma 4.1]), one has the following *isomorphisms of locally m-convex algebras*

$$\Gamma(\mathcal{O}(X,\xi)) = \Gamma(\mathcal{O}_X \hat{\otimes} \xi) = \mathcal{O}_X \underset{\tau}{\hat{\otimes}} \Gamma(\xi) = \mathcal{O}(X, \Gamma(\xi)) \tag{5.16}$$

($\tau \equiv \pi = \varepsilon$), concerning the global section functor "Γ" and "$\mathcal{O}$" (holomorphic functions). Thus the corresponding *spectrum bundle of (5.15)* is given by

$$\mathcal{M}(\mathcal{O}(X,\xi)) = X \times \mathcal{M}(\xi), \tag{5.17}$$

within a *fibre bundle isomorphism* (cf. remarks after Theorem 5.2). This is valid for suitable complex manifolds and locally m-convex algebra bundles, in order to take the next *homeomorphisms* (cf. (5.16), [6: Chapter VII, Lemma 3.1]). Indeed, one has

$$\begin{aligned}\mathcal{M}(\mathcal{O}(X,\xi)) &= \mathcal{M}(\mathcal{O}_X \hat{\otimes} \xi) = \mathcal{M}(\mathcal{O}_X) \times \mathcal{M}(\xi) \\ &= X \times \mathcal{M}(\xi)\end{aligned} \tag{5.18}$$

(cf. Theorem 5.2)

5.(4). *The bundle $L^1(G,\xi)$.* Let G be a *locally compact (abelian) group* and $L^1(G)$ the *group algebra* of G (cf. [6: Chapter VII, § 4]). If ξ is a locally convex algebra bundle, then

$$L^1(G,\xi) := L^1(G) \hat{\otimes} \xi \tag{5.19}$$

(cf. (5.1)) is the *group algebra bundle of G, relative to ξ*. In this case one gets the corresponding algebra of sections.

$$\Gamma(L^1(G,\xi)) = L^1(G) \underset{\tau}{\hat{\otimes}} \Gamma(\xi) = L^1(G, \Gamma(\xi)), \tag{5.20}$$

($\tau \equiv \pi = \varepsilon$) under suitable conditions (cf. the hypotheses of Theorem 5.1, relation (5.2) and also [6: Chapter XI, (5.14)]). Here the equalities mean *isomorphisms of locally convex algebras*. Moreover, the corresponding *spectrum bundle* of (5.19) is of the form

$$\mathcal{M}(L^1(G, \xi)) = \hat{G} \times \mathcal{M}(\xi) \ , \tag{5.21}$$

within a *fibre bundle isomorphism*, with $\hat{G}$ the *character group* of G (cf [6: Chapter VII, (4.11)]). Indeed, the last relation is valid under suitable conditions (cf., for instance, the hypotheses of Theorem 5.2, relations (5.2),(5.3) and also [6: Chapter VII, Theorem 4.1]), since

$$\begin{aligned} \mathcal{M}(L^1(G, \xi)) &= \mathcal{M}(L^1(G)\hat{\underset{\tau}{\otimes}}\Gamma(\xi)) = \mathcal{M}(L^1(G)) \times \mathcal{M}(\Gamma(\xi)) \\ &= \hat{G} \times \mathcal{M}(\xi) \end{aligned} \tag{5.22}$$

within *homeomorphisms of the topological spaces concerned*.

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