Embeddings and Expansions

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Abstract

In this paper we sketch a general theory of embeddings for geometries with string diagrams, focusing on their hulls. An affine-like geometry, which we call expansion, is associated to every embedding. As we shall prove, the universal cover of the expansion of an embedding is the expansion of the hull of that embedding. Some applications of this theorem are given.

1 Introduction

1.1 Aims and organization of this paper

I shall sketch a general theory of embeddings for geometries belonging to string diagrams, focusing on hulls of embeddings and on the problem of extending an embedding from the point-line system of a geometry Γ to the whole of Γ. In order to cover as many facts as I can, I don't require the codomain of an embedding to be a projective space, allowing the subgroup lattice of any group to be a feasible codomain. This abstract approach will reward us with simplifying some constructions and the solution of some problems and will provide a framework where both projective embeddings as defined by Ronan [30] and representation groups in the sense of Ivanov and Shpectorov [18] can be placed quite naturally.

The idea of embedding a geometry in a group rather than in a projective space is mainly motivated by the investigation of non-abelian representations of P- and T-geometries (see Ivanov and Shpectorov [18]; also Ivanov, Pasechnik and Shpectorov [19], Ivanov [16]). However, I am not going to survey embeddings of P- and T-geometries in this paper. The reader is referred to [18] for that. I have preferred to focus on classical objects, as projective spaces, polar spaces, grassmannians and dual polar spaces.

Received by the editors February 2002.
Communicated by H. Van Maldeghem.

The paper is organized as follows. In the rest of the Introduction we state some notation and terminology for geometries with string diagrams and we recall the definition of the geometry far from a flag of a building. Embeddings and morphisms of embeddings are defined in Section 2. An affine-like structure, which we call *expansion*, is associated to every embedding and the functor sending every embedding to its expansion also sends morphisms of embeddings to coverings of expansions. A number of basic facts on embeddings and expansions are proved in Section 2. Many of them generalize results stated by Ivanov and Shpectorov [18] for representations of geometries with three points per line.

Section 3 is devoted to hulls. Every embedding ε admits a *hull*, which is universal in the class of embeddings that dominate ε. We prove that the universal cover of the expansion of ε is the expansion of the hull of ε, thus generalizing a result of Ivanov and Shpectorov [18, 2.5.1]. In Section 4 we investigate the following problem: Given an embedding ε of a geometry Γ of rank at least 3, let ε_0 be the embedding induced by ε on the point-line system Σ of Γ. When does it happen that the hull of ε induces on Σ the hull of ε_0?

In Sections 5, 6,..., 10 we apply some of the results proved in Sections 2, 3 and 4 to investigate hulls of embeddings of projective spaces, polar spaces, grassmannians of projective spaces, dual polar spaces and half-spin geometries. In particular, in Sections 5 and 6 we study embeddings of projective geometries, stating some general results in Section 5, then focusing on two particular embeddings in Section 6 which, as we will show in Section 9, are involved in projective embeddings of dual polar spaces of symplectic and hermitian type. We call them (*plain* and *twisted*) *tensor embeddings*, after the way in which they are constructed (but plain tensor embeddings might be called 'Veronesean embeddings' as well, as the image of such an embedding is a Veronesean quadric).

Section 7 is devoted to polar spaces. In that section we prove that the hull of a projective embedding of a classical polar space Γ of rank $n \geq 2$ embeds Γ in the unipotent radical of a point-stabilizer in $\mathrm{Aut}(\Pi)$, for a polar space Π of the same kind as Γ but of rank $n+1$. As a by-product, we obtain a fairly easy proof of the well known fact that a projective embedding of a polar space in odd characteristic is not the image of any larger projective embedding. We also prove that a projective embedding $\varepsilon : \Gamma \to PG(V)$ of a polar space Γ is its own hull if and only if $\varepsilon(\Gamma)$ is a quadric of $PG(V)$.

In Section 8 we prove that the natural embedding of the line-grassmannian of a projective space is its own hull. We obtain a similar result for the line-grassmannian of an affine space, provided that the underlying field is not $GF(2)$. Embeddings of dual polar spaces are discussed in Section 9, with particular emphasis on cases of rank 3. In particular, we prove that the spin embedding of the dual of $\mathcal{Q}_6(q)$ is its own hull. We also describe the hulls of the embeddings of the duals of $\mathcal{W}_5(q)$ and $\mathcal{H}_5(q^2)$ in $PG(13, q)$ and $PG(19, q)$, but assuming that q is prime. In Section 10 we consider the projective embedding of the half-spin geometry of the building of type D_n proving that, when $n \leq 5$, that embedding is its own hull.

1.2 Terminology and notation for geometries

As in [26], all geometries are residually connected and firm by definition. Given a geometry Γ, we write $X \in \Gamma$ to say that X is an element of Γ. Given $X \in \Gamma$, we denote the type of X by $t(X)$ and its residue by $\mathrm{Res}_\Gamma(X)$ (also $\mathrm{Res}(X)$ when no ambiguity arises). Given a subset $J \neq \emptyset$ of the set of types of Γ, the *J-truncation* of Γ is the geometry obtained from Γ by removing all elements of type $j \notin J$.

We are not interested in non-type-preserving automorphisms in this paper. Accordingly, we denote by $Aut(\Gamma)$ the group of type-preserving automorphisms of Γ. We call them just *automorphisms* of Γ.

As in [26], given two geometries Γ and Δ of rank n overthe same set of types and a positive integer $m < n$, an *m-covering* from Γ to Δ is a type-preserving morphism $\varphi : \Gamma \to \Delta$ such that, for every flag F of Γ of corank m, the restriction of φ to $\mathrm{Res}_\Gamma(F)$ is an isomorphism to $\mathrm{Res}_\Delta(\varphi(F))$. Accordingly, if $\varphi : \Gamma' \to \Gamma$ is an m-covering, we say that Γ' is an *m-cover* of Γ and Γ is an *m-quotient* of Γ'. If Γ is its own universal m-cover, then we say that Γ is *m-simply connected*. We call the $(n-1)$-coverings just *coverings*, for short. Accordingly, the *universal cover* of a geometry Γ of rank n is its $(n-1)$-universal cover and Γ is said to be *simply connected* if it is $(n-1)$-simply connected.

1.3 Poset-geometries

We say that a geometry Γ is a *poset-geometry* when its set of types is equipped with a total ordering $\leq$ such that, for any three elements X, Y, Z of Γ, if $t(X) \leq t(Y) \leq t(Z)$ and Y is incident with both X and Z, then X is incident to Z. Given two elements X, Y of a poset-geometry Γ, we write $X < Y$ (respectively $X \leq Y$) when X and Y are incident and $t(X) < t(Y)$ (resp. $t(X) \leq t(Y)$).

We will always take the integers $0, 1, ..., n-1$ as types for a poset-geometry of rank n, ordered in the natural way. The elements of type 0 and 1 are called *points* and *lines*, respectively. Two points are said to be *collinear* if they belong to the same line. The relation 'being collinear' defines a graph on the set of points of Γ, called the *collinearity graph* of Γ. The 2-elements are also called *planes*, but different words are used in special contexts instead of 'planes' (as *quads*, for instance, when dealing with dual polar spaces).

We denote by $P(\Gamma)$ and $L(\Gamma)$ the set of points and the set of lines of Γ. Similarly, given an element $X \in \Gamma$, we denote by $P(X)$ the set of points $p \leq X$ and by $L(X)$ the set of lines incident to X, with the convention that $P(p) = \{p\}$ and $L(L) = \{L\}$ for every point p and every line L. Clearly, if $X \leq Y$, then $P(X) \subseteq P(Y)$.

Given an element A of type $t(A) > 0$ (respectively, $t(A) < n-1$) the *lower* (*upper*) *residue* of A is the poset-geometry induced by Γ on the set of elements $X < A$ (resp. $X > A$). We shall denote the lower and upper residues of A by $\mathrm{Res}^-(A)$ and $\mathrm{Res}^+(A)$.

The *irreducible* poset-geometries, which do not split as a direct sum of smaller geometries, are those belonging to a string diagram, possibly of rank 1.

1.4 Subgeometries far from a flag of a building

In this subsection Γ is a thick building of connected spherical type and rank at least 2. It is well known that, given a flag $F \neq \emptyset$ and a chamber C of Γ, there is a unique chamber $C_F \in \mathrm{Res}(F)$ at minimal distance from C (Tits [35]). We denote the distance between C and C_F by $d(C, F)$. For every nonempty flag X, the *distance* $d(X, F)$ from X to F is the minimal distance $d(C, F)$ from F to a chamber $C \supseteq X$. We say that a flag X is *far* from F if $d(X, F)$ is maximal, compatibly with the types of F and X. We denote by $\mathrm{Far}_\Gamma(F)$ the substructure of Γ formed by the elements far from F, with the incidence relation inherited from Γ, but rectified as follows: two elements $X, Y \in \mathrm{Far}_\Gamma(F)$ are incident in $\mathrm{Far}_\Gamma(F)$ if and only if they are incident in Γ and the flag $\{X, Y\}$ is far from F.

It is known that $\mathrm{Far}_\Gamma(F)$ is residually connected, except for a few cases defined over $GF(2)$ (Blok and Brouwer [4]), but none of those exceptional cases will be met in this paper.

2 Definitions and basics

2.1 Embeddings

An *embedding* $\varepsilon : \Gamma \to G$ of a poset-geometry Γ in a group G is an injective mapping ε from the set of elements of Γ to the set of proper non-trivial subgroups of G such that:

(E1) for $X, Y \in \Gamma$, we have $\varepsilon(X) \leq \varepsilon(Y)$ if and only if $X \leq Y$;

(E2) $\varepsilon(X) = \langle \varepsilon(p) \rangle_{p \in P(X)}$ for every $X \in \Gamma$;

(E3) $G = \langle \varepsilon(p) \rangle_{p \in P(\Gamma)}$.

We call the group G the *codomain* of ε and we denote it by *cod*(ε). When G is commutative, we say that the embedding ε is *abelian*. In particular, if G is the additive group of a vector space V defined over a given division ring K and $\varepsilon(p)$ is a linear subspace of V for every $p \in P(\Gamma)$, then we say that ε is a K-*linear embedding* of Γ in V (also a *linear embedding*, for short). In this case we slightly change the previous conventions, calling V the *codomain* of ε, thus writing $\mathrm{cod}(\varepsilon) = V$ and $\varepsilon : \Gamma \to V$.

Proposition 2.1. *A geometry Γ admits an embedding if and only if it satisfies the following property:*

(PS) (Point-Set Property) *for any two elements $X, Y \in \Gamma$, if $P(X) \subseteq P(Y)$, then $X \leq Y$.*

Furthermore, if Γ admits an embedding, then it also admits a linear embedding.

Proof. Let $\varepsilon : \Gamma \to G$ be an embedding and suppose $P(X) \subseteq P(Y)$. Then $\varepsilon(X) \leq \varepsilon(Y)$ by (E2), whence $X \leq Y$ by (E1). Conversely, assume (PS). Given a K-vector space V of dimension $\dim(V) = |P(\Gamma)|$ and a basis $B = \{b_p\}_{p \in P(\Gamma)}$ of V, define $\varepsilon(X) := \langle b_p \rangle_{p \in P(X)}$ for every $X \in \Gamma$. The function ε defined in this way is a linear embedding of Γ in V. ∎

Note that (PS) forces Γ to be irreducible. Accordingly, henceforth, only irreducible geometries will be considered.

2.2 Projective and locally projective embeddings

Lax and full projective embeddings. Suppose that Γ has rank at least 2 and let $\varepsilon : \Gamma \to V$ be a K-linear embedding. Following Van Maldeghem [37], if $\dim(\varepsilon(p)) = 1$ for all points $p \in P(\Gamma)$ and $\dim(\varepsilon(L)) = 2$ for every line L of Γ, then we say that ε is a *lax projective* embedding *defined* over K (a *lax K-projective* embedding, for short) and we write $\varepsilon : \Gamma \to PG(V)$. If furthermore $\varepsilon(L) = \cup_{p\in P(L)}\varepsilon(p)$ for every line L of Γ, then we say that ε is *full*.

Note 2.1. Only geometries of rank 2 are considered by Van Maldeghem [37] and Ronan [30]. So, if we strictly followed [37] or [30], we should consider the point-line system $(P(\Gamma), L(\Gamma))$ of Γ rather than Γ itself. Clearly, every projective embedding of Γ as defined above induces a projective embedding of $(P(\Gamma), L(\Gamma))$, but the converse is false in general (see Subsection 2.5, Example 2.1).

Locally projective embeddings. Still with Γ of rank at least 2, let $\varepsilon : \Gamma \to G$ be an embedding of Γ and let K be a division ring. Without assuming that ε is linear, suppose that two families $\{V(p)\}_{p\in P(\Gamma)}$ and $\{V(L)\}_{L\in L(\Gamma)}$ of K-vector spaces are given such that:

(P1) for every point p, $\dim(V(p)) = 1$ and $\varepsilon(p)$ is the additive group of $V(p)$;

(P2) for every line L, $\dim(V(L)) = 2$ and $\varepsilon(L)$ is the additive group of $V(L)$;

(P3) for every point p and every line L, if $p < L$ then $V(p)$ is a subspace of $V(L)$.

Then we say that ε is a *lax locally K-projective* embedding. If furthermore,

(P4) for every line L, $\{V(p)\}_{p\in P(L)}$ is the family of all 1-dimensional linear subspaces of $V(L)$,

then we say that ε is a *full locally K-projective* embedding. Note that, in view of (E1) and (P2), if Γ admits a lax locally K-projective embedding ε, then no two distinct lines of Γ have more than one point in common. If furthermore ε is full, then all lines of Γ have $|K| + 1$ points.

Note 2.2. Representations as considered by Ivanov [16] are just full locally $GF(p)$-projective embeddings, for a prime p. In particular, representations groups as defined by Ivanov an Shpectorov [18] are full locally $GF(2)$-projective embeddings.

2.3 Expansions

Given an embedding $\varepsilon : \Gamma \to G$ of a poset geometry Γ in a group G, we define a poset-geometry $\mathrm{Exp}(\varepsilon)$ of rank $n + 1$ as follows: The points of $\mathrm{Exp}(\varepsilon)$ are the elements of G and, for $i = 1, 2, ..., n$, the i-elements of $\mathrm{Exp}(\varepsilon)$ are the right cosets

$g{\cdot}\varepsilon(X)$, for $g \in G$ and $X \in \Gamma$ with $t(X) = i-1$. The incidence relation is the natural one, namely inclusion between cosets and between elements and cosets. We call $\mathrm{Exp}(\varepsilon)$ the *expansion* of Γ to G via ε, also the *expansion* of ε, for short.

It is not difficult to check that $\mathrm{Exp}(\varepsilon)$ is indeed a poset-geometry. In particular, the residual connectedness of $\mathrm{Exp}(\varepsilon)$ follows from conditions (E2) and (E3) and from the residual connectedness of Γ (see Propositions 2.2, 2.3 and 2.4, to be stated below).

Proposition 2.2. *Given two distinct points $a, b \in G$ of $\mathrm{Exp}(\varepsilon)$, let $d(a, b)$ be their distance in the collinearity graph of $\mathrm{Exp}(\varepsilon)$. Then,*

(1) *$d(a, b)$ is the minimal length of an m-tuple $(p_1, p_2, ..., p_m)$ of points of Γ such that $b^{-1}a \in \varepsilon(p_1)\varepsilon(p_2)...\varepsilon(p_m)$;*

(2) *if $d(a, b) = 1$, then the number of lines of $\mathrm{Exp}(\varepsilon)$ containing both a and b is equal to the number of points $p \in P(\Gamma)$ for which $b^{-1}a \in \varepsilon(p)$.*

Proof. (2) is obvious and (1) follows from this remark: A sequence $a_0, a_1, ..., a_m$ of points of $\mathrm{Exp}(\varepsilon)$ is a path in the collinearity graph of $\mathrm{Exp}(\varepsilon)$ if and only if, for every $i = 1, 2, ..., m$, $a_{i-1}^{-1}a_i \in \varepsilon(p_i)$ for some $p_i \in P(\Gamma)$. ■

Proposition 2.3. *For every $g \in G$, the function sending $g{\cdot}\varepsilon(X)$ to $X \in \Gamma$ is an isomorphism from $\mathrm{Res}_{Exp(\varepsilon)}(g)$ to Γ.* ■

For every element $X \in \Gamma$ of type $t(X) > 0$, the mapping ε induces an embedding ε_X of $\mathrm{Res}^-_\Gamma(X)$ in $\varepsilon(X)$. The following are obvious:

Proposition 2.4. *Regarding $\varepsilon(X)$ as an element of $\mathrm{Exp}(\varepsilon)$, its lower residue $\mathrm{Res}^-_{\mathrm{Exp}(\varepsilon)}(\varepsilon(X))$ is the expansion $\mathrm{Exp}(\varepsilon_X)$ of $\mathrm{Res}^-_\Gamma(X)$ to the group $\varepsilon(X)$ via ε_X. Furthermore, for any $g \in G$, the left multiplication by g^{-1} induces an isomorphism from $\mathrm{Res}^-_{\mathrm{Exp}(\varepsilon)}(g{\cdot}\varepsilon(X))$ to $\mathrm{Exp}(\varepsilon_X)$.* ■

Proposition 2.5. *The group G, in its action on itself by left multiplication, is a group of automorphisms of $\mathrm{Exp}(\varepsilon)$, regular on the set of points of $\mathrm{Exp}(\varepsilon)$. Furthermore, for every $X \in \Gamma$ of type $t(X) > 0$ and every $g \in G$, the stabilizer $g\varepsilon(X)g^{-1}$ of $g{\cdot}\varepsilon(X)$ in G acts transitively (whence, regularly) on the set of points of $g{\cdot}\varepsilon(X)$.* ■

Clearly, the Point-Set Property (PS) holds in $\mathrm{Exp}(\varepsilon)$.

Proposition 2.6. *Let Δ be a poset-geometry of rank at least 2, satisfying the Point-Set Property (PS). Then the following are equivalent:*

(1) *there exist a poset-geometry Γ, a group G and an embedding $\varepsilon : \Gamma \to G$ such that $\Delta \cong \mathrm{Exp}(\varepsilon)$;*

(2) *$\mathrm{Aut}(\Delta)$ admits a subgroup G acting regularly on $P(\Delta)$ and such that, for every $U \in \Delta$, the stabilizer G_U of U in G acts transitively on $P(U)$.*

Proof. By Proposition 2.5, (1) implies (2). Conversely, assume (2). Pick a point $p \in P(\Delta)$. For any two elements U, W of $\mathrm{Res}(p)$, we have $U \leq W$ if and only if $G_U \leq G_W$. (Recall that, according to (2), G_U and G_W act regularly on $P(U)$ and $P(W)$.) So, the mapping ε from $\Gamma := \mathrm{Res}(p)$ to the subgroup lattice of G sending $U \in \mathrm{Res}(p)$ to $\varepsilon(U) := G_U$, satisfies (E1). We shall prove that ε satisfies (E3). Let $g \in G$. By connectedness, the collinearity graph of Δ contains a path $x_0 = p, x_1, x_2, ..., x_k = g(p)$. For $i = 1, 2, ..., k$, let L_i be a line through x_{i-1} and x_i. By the transitivity of G_{L_i} on $P(L_i)$, there exists an element $f_i \in G_{L_i}$ sending x_{i-1} to x_i. Put $g_i := f_i ... f_2 f_1$, $M_1 := L_1$ and $M_i := f_{i-1}^{-1}(L_i)$ for $i = 2, 3, ..., k$. Then $f_i \in f_{i-1} G_{M_i} f_{i-1}^{-1}$, $g_k = g$ by the regularity of G and, by an easy inductive argument, $g_h \in \langle G_{M_i} \rangle_{i=1}^{h}$ for $h = 1, 2, ...k$. Therefore $g = g_k \in \langle G_{M_i} \rangle_{i=1}^{k} = \langle \varepsilon(M_i) \rangle_{i=1}^{k}$. So, ε satisfies (E3). Property (E2) can be proved in a similar way. Thus, ε is an embedding of Γ in G. Obviously, $\Delta \cong \mathrm{Exp}(\varepsilon)$. ∎

Note 2.3. A very general construction of expansion is given by Buekenhout, Huybrechts and Pasini [6] in the context of an axiomatic theory of parallelisms. Our expansions are a special case of that construction.

2.4 Morphisms

Given two embeddings $\varepsilon : \Gamma \to G$ and $\eta : \Gamma \to F$, a *morphism* from ε to η is a homomorphism $f : G \to F$ such that, for every $X \in \Gamma$, the restriction of f to $\varepsilon(X)$ is an isomorphism to $\eta(X)$. If furthermore $f : G \to F$ is an isomorphism, then we say that f is an *isomorphism* from ε to η. If a morphism exists from ε to η, then we say that η is a *homorphic image* of ε (also, an *image* of ε, for short) and that ε *dominates* η. If there is an isomorphism from ε to η, then we say that ε and η are *isomorphic* and we write $\varepsilon \cong \eta$.

Given a morphism $f : \varepsilon \to \eta$, for every $X \in \Gamma$ the homomorphism f maps the right cosets of $\varepsilon(X)$ in G onto right cosets of $\eta(X)$ in F. Accordingly, f defines a morphism $\mathrm{Exp}(f) : \mathrm{Exp}(\varepsilon) \to \mathrm{Exp}(\eta)$.

Proposition 2.7. *The morphism* $\mathrm{Exp}(f)$ *is a covering. Furthermore,* $\mathrm{Exp}(f)$ *is an isomorphism if and only if f is an isomorphism.* ∎

The coverings of expansions that, modulo isomorphisms, arise from morphisms of embeddings are characterized in the following proposition.

Proposition 2.8. *Let* $\varphi : \widehat{\Delta} \to \Delta := \mathrm{Exp}(\varepsilon)$ *be a covering. Then the following are equivalent:*

(1) *there exists an embedding* $\hat{\varepsilon} : \Gamma \to \widehat{G}$ *and a morphism of embeddings* $f : \hat{\varepsilon} \to \varepsilon$ *such that* $\widehat{\Delta} \cong \mathrm{Exp}(\hat{\varepsilon})$ *and* $\varphi = \mathrm{Exp}(f)\alpha$ *for an isomorphism* $\alpha : \widehat{\Delta} \xrightarrow{\cong} \mathrm{Exp}(\hat{\varepsilon})$*;*

(2) *the group of deck transformations of* φ *acts transitively* (*whence, regularly*) *on each of the fibers of* φ *and* G*, regarded as a subgroup of* $\mathrm{Aut}(\Delta)$*, lifts through* φ *to a subgroup of* $\mathrm{Aut}(\widehat{\Delta})$.

Proof. Assume (1) and let $K := \mathrm{Ker}(f)$. Then K is the group of deck transformations of φ and acts regularly on each of the fibers of φ. Furthermore $\widehat{G}$, regarded as a subgroup of $\mathrm{Aut}(\widehat{\Delta})$, is the lifting of G to $\widehat{\Delta}$ through φ. So, (2) holds.

Conversely, assume (2). Let $\widehat{G}$ be the lifting of G to $\widehat{\Delta}$ and $D(\varphi)$ be the group of deck transformations of φ. We recall that $\widehat{G}$ contains $D(\varphi)$ as a normal subgroup. As G is regular on $P(\Delta)$ (Proposition 2.5) and, by assumption, $D(\varphi)$ acts regularly on each of the fibers of φ, $\widehat{G}$ acts regularly on $P(\widehat{\Delta})$. Pick $a \in \varphi^{-1}(1)$, where $1 \in G$ is regarded as a point of Δ. For every element $\widehat{U} \in \mathrm{Res}_{\widehat{\Delta}}(a)$ and every point $x < \widehat{U}$, let g be the element of G ($\leq \mathrm{Aut}(\Delta)$) sending 1 to $\varphi(x)$ and stabilizing $U := \varphi(\widehat{U})$. (Such an element g exists and is unique, by Proposition 2.5.) Let $\hat{g}$ be a lifting of g. Modulo replacing $\hat{g}$ with $\delta\hat{g}$ for a suitable $\delta \in D(\varphi)$, we may assume that $\hat{g}(a) = x$. Hence $\hat{g}$ stabilizes $\widehat{U}$. (Note that, as Property (PS) is preserved when taking covers and (PS) holds in Δ, $\widehat{\Delta}$ satisfies (PS).) We can now apply Proposition 2.6, thus obtaining that the function $\hat{\varepsilon}$ sending $\widehat{U} > a$ to its stabilizer in $\widehat{G}$ is an embedding of $\Gamma \cong \mathrm{Res}_{\widehat{\Delta}}(a)$ in $\widehat{G}$. Let f be the projection of $\widehat{G}$ onto G. Then $\mathrm{Ker}(f) = D(\varphi)$. However, $D(\varphi)$ acts semi-regularly on the set of elements of $\widehat{\Delta}$. Hence $\mathrm{Ker}(f) \cap \hat{\varepsilon}(\widehat{U}) = 1$. Therefore, f is a morphism from $\hat{\varepsilon}$ to ε and $\varphi = \mathrm{Exp}(f)\alpha$, where $\alpha : \widehat{\Delta} \overset{\cong}{\to} \mathrm{Exp}(\hat{\varepsilon})$ sends $\hat{g}(\widehat{U})$ to the coset $\hat{g}\widehat{G}_{\widehat{U}}$ of the stabilizer $\widehat{G}_{\widehat{U}}$ of $\widehat{U}$ in $\widehat{G}$, for $\hat{g} \in \widehat{G}$ and $\widehat{U} > a$. ∎

The linear case. Suppose now that ε and η are K-linear for a given division ring K. Let $V := \mathrm{cod}(\varepsilon)$ and $W := \mathrm{cod}(\eta)$. According to the previous definition, a morphism from ε to η is a homomorphism f from the additive group of V to the additive group of W such that $f(\varepsilon(X)) = \eta(X)$ and $\mathrm{Ker}(f) \cap \varepsilon(X) = 0$ for every $X \in \Gamma$. Note that f need not be a semilinear mapping of vector spaces. If f is a semilinear (in particular, linear) mapping from V to W, then we say that the morphism $f : \varepsilon \to \eta$ is *linear*. If $\varepsilon \cong \eta$ and there exists a linear isomorphism from ε to η, then we say that ε and η are *linearly isomorphic* and we write $\varepsilon \cong_{\mathrm{lin}} \eta$.

Note 2.4. We warn that, usually, only linear morphisms are considered in the literature on projective embeddings.

2.5 Extensions of embeddings

Assume that Γ has rank $n \geq 2$ and, for an integer $m = 2, 3, ..., n-1$, let Σ be the $\{0, 1, ..., m-1\}$-truncation of Γ. Suppose that Σ admits an embedding $\varepsilon_0 : \Sigma \to G$. For every $X \in \Gamma$, put $\varepsilon(X) := \langle \varepsilon_0(p) \rangle_{p \in P(X)}$. By (E2) on ε_0, if $t(X) < m$ then $\varepsilon(X) = \varepsilon_0(X)$. So, ε is an embedding if and only if it satisfies (E1). (Indeed ε satisfies (E2) by definition, (E3) holds since it holds for ε_0 and (E1) forces ε to be injective and $\varepsilon(X)$ to be different from G.) Clearly, if ε is an embedding, then it is the unique embedding of Γ inducing ε_0 on Σ. We call it the *extension* of ε_0 to Γ and we say that ε_0 *extends* to Γ.

Examples of non-extensible embeddings are very easy to construct: Assume that Γ has rank 3, put $\Sigma := (P(\Gamma), L(\Gamma))$ and suppose that no two lines of Σ have the same points. Then Σ is embeddable, by Proposition 2.1. However, Property (PS) might fail to hold in Γ. If that is the case, then Γ is not embeddable and, therefore, none of the embeddings of Σ extends to Γ. But even if Γ satisfied (PS), some of the embeddings of Σ might not extend to Γ, as in the following example.

Example 2.1. Given an ovoid O of $PG(3, q)$, let Γ be the geometry of points, pairs and triples of points of O, with the natural incidence relation. A lax projective embedding ε_0 of $(P(\Gamma), L(\Gamma))$ in $PG(3, q)$ is implicit in this definition but, when $q > 2$, ε_0 does not extend to Γ. Indeed, any two triples contained in the same plane of $PG(3, q)$ span that plane, and this contradicts (E1).

2.6 Reducible embeddings

Given an embedding $\varepsilon : \Gamma \to G$, we put $\mathrm{Rad}(\varepsilon) := \cap_{p\in P(\Gamma)}\varepsilon(p)$ and denote by $\mathrm{Ker}(\varepsilon)$ the maximal normal subgroup of G contained in $\mathrm{Rad}(\varepsilon)$. We say that ε is *reducible* if $\mathrm{Ker}(\varepsilon) \neq 1$. If ε is reducible and $K \leq \mathrm{Ker}(\varepsilon)$ is normal in G, then the mapping ε/K sending every element $X \in \Gamma$ to $\varepsilon(X)/K$ is an embedding of Γ in G/K. We call ε/K the *quotient* of ε by K.

Suppose that $\mathrm{Rad}(\varepsilon) \neq 1$ and let R be a non-trivial subgroup of $\mathrm{Rad}(\varepsilon)$. Then $\{gR\}_{g\in G}$ is a partition of the set of points of $\mathrm{Exp}(\varepsilon)$ and, for every element $U \in \mathrm{Exp}(\varepsilon)$ of type $t(U) > 0$, $U/R := \{gR\}_{g\in U}$ is a partition of U. We define a geometry $\mathrm{Exp}(\varepsilon)/R$, which we call the *quotient* of $\mathrm{Exp}(\varepsilon)$ over R, by taking the right cosets of R as points and the partitions U/R as elements of positive type.

The function π_R sending every point of $\mathrm{Exp}(\varepsilon)$ to the right coset of R containing it and every $U \in \mathrm{Exp}(\varepsilon)$ of positive type to U/R, is a morphism of geometries from $\mathrm{Exp}(\varepsilon)$ to $\mathrm{Exp}(\varepsilon)/R$. The morphism π_R is not a covering but, for every point g of $\mathrm{Exp}(\varepsilon)$, it induces an isomorphism from $\mathrm{Res}_{\mathrm{Exp}(\varepsilon)}(g)$ to the residue of $gR = \pi_R(g)$ in $\mathrm{Exp}(\varepsilon)/R$. Clearly, if $R \trianglelefteq G$, then $\mathrm{Exp}(\varepsilon)/R \cong \mathrm{Exp}(\varepsilon/R)$.

Proposition 2.9. *Let ε and η be two embeddings of Γ and, for $R \leq \mathrm{Rad}(\varepsilon)$ and $S \leq \mathrm{Rad}(\eta)$, suppose that $|R| = |S|$ and $\mathrm{Exp}(\varepsilon)/R \cong \mathrm{Exp}(\eta)/S$. Then $\mathrm{Exp}(\varepsilon) \cong \mathrm{Exp}(\eta)$.*

Proof. Given an isomorphism α from $\mathrm{Exp}(\varepsilon)/R$ to $\mathrm{Exp}(\eta)/S$, for every right coset C of R, choose a bijection α_C from C to the right coset $\alpha(C)$ of S and, for every $g \in C$, put $\beta(g) = \alpha_C(g)$. Then β is an isomorphism from $\mathrm{Exp}(\varepsilon)$ to $\mathrm{Exp}(\eta)$. ∎

Example 2.2 (Parabolic systems). Suppose that Γ is a simplex and that $\varepsilon(p_1) \cap \varepsilon(p_2) = \mathrm{Rad}(\varepsilon)$ for any two distinct points $p_1, p_2 \in P(\Gamma)$. Then $\mathcal{P} := (\varepsilon(p))_{p\in P(\Gamma)}$ is a parabolic system over the set of types $I := P(\Gamma)$ and the group $B := \mathrm{Rad}(\varepsilon)$ is its Borel subgroup. $\mathrm{Exp}(\varepsilon)/B$ is the chamber system associated to $\mathcal{P}$, equipped with its cells.

2.7 Embeddings induced on point-residues

Given an embedding $\varepsilon : \Gamma \to G$ of a geometry Γ of rank at least 2 and a point $p \in P(\Gamma)$, we put $G_p := \langle\varepsilon(L)\rangle_{L\in L(p)}$ and, for $X \in \mathrm{Res}(p)$, $\varepsilon(X)_p := \langle\varepsilon(L)\rangle_{L\in L(p)\cap L(X)}$.

Proposition 2.10. *Suppose that* $\mathrm{Res}(p)$ *satisfies the Point-Set Property* (PS)*, namely: For any two elements* $X, Y > p$*, if* $L(p) \cap L(X) \subseteq L(p) \cap L(Y)$*, then* $X \leq Y$*. Then the mapping* ε_p *sending* $X > p$ *to* $\varepsilon(X)_p$ *is an embedding of* $\mathrm{Res}(p)$ *in* G_p *and* $\varepsilon(p) \leq \mathrm{Rad}(\varepsilon_p)$*.*

Proof. For $X, Y \in \mathrm{Res}(p)$, let $\varepsilon_p(X) \leq \varepsilon_p(Y)$. Then $\varepsilon_p(L) \leq \varepsilon_p(Y)$ for every line L with $p < L \leq X$. Hence $\varepsilon(L) = \varepsilon(L)_p \leq \varepsilon(Y)_p \leq \varepsilon(Y)$. Therefore $L \leq Y$ by (E1) on ε. So, all lines of X on p are incident to Y. Consequently, $X \leq Y$ by (PS) in $\mathrm{Res}(p)$. ■

We call ε_p the embedding of $\mathrm{Res}(p)$ *induced* by ε. When $\varepsilon(p) \trianglelefteq G_p$, we can also consider the quotient $\varepsilon_p/\varepsilon(p)$, which we call the *reduction* of ε_p. (Note that, however, $\varepsilon_p/\varepsilon(p)$ might still be reducible.)

Note 2.5. The quotient $\mathrm{Exp}(\varepsilon_p)/\varepsilon(p)$ is a shrinking of $\mathrm{Exp}(\varepsilon)$, in the meaning of Pasini and Wiedorn [28] (see also Stroth and Wiedorn [34]).

2.8 The rank 1 case

A geometry of rank 1 is just a set S of size at least two. An embedding of such a geometry in a group G is a bijection $\varepsilon : S \stackrel{1\text{–}1}{\to} \mathcal{P}$ from S to a collection $\mathcal{P}$ of proper non-trivial subgroups of G such that $G = \langle P \rangle_{P \in \mathcal{P}}$ and no member of $\mathcal{P}$ contains any other member of $\mathcal{P}$. We recall that $\mathcal{P}$ is said to be a *partition* of G if $\{P \setminus \{1\}\}_{P \in \mathcal{P}}$ is a partition of the set $G \setminus \{1\}$. The statements gathered in the next proposition immediately follow from Proposition 2.2.

Proposition 2.11. *With* S*,* ε *and* $\mathcal{P}$ *as above, the geometry* $\mathrm{Exp}(\varepsilon)$ *is a partial linear space if and only if the members of* $\mathcal{P}$ *have trivial mutual intersections. It is a* 2*-design if and only if* $\cup_{P \in \mathcal{P}} P = G$*. Hence* $\mathrm{Exp}(\varepsilon)$ *is a linear space if and only if* $\mathcal{P}$ *is a partition of* G*.* ■

When $\mathcal{P}$ is a partition of G, $\mathrm{Exp}(\varepsilon)$ is a translation André structure [1]. In particular, when any two distinct members of $\mathcal{P}$ commute and generate G, then $\mathrm{Exp}(\varepsilon)$ is an affine translation plane. In the very special case where $\mathcal{P}$ is the collection of 1-dimensional subspaces of $V(2, K)$, then $\mathrm{Exp}(\varepsilon) = AG(2, K)$.

When any two distinct members of $\mathcal{P}$ intersect trivially, commute and generate G, then $\mathrm{Exp}(\varepsilon)$ is a net. In particular, for $|\mathcal{P}| = 2$ we get a grid. When $G = \oplus_{P \in \mathcal{P}} P$, then $\mathrm{Exp}(\varepsilon)$ is a Hamming cube (a grid when $|\mathcal{P}| = 2$). When G is the additive group of a vector space V and $\mathcal{P}$ a collection of proper non-trivial subspaces of V such that $V = \langle \mathcal{P} \rangle$, the embedding ε is linear. Some examples of this situation are described below. In each of them $V = V(n, q)$ for a prime power q and all members of $\mathcal{P}$ have the same dimension d.

Example 2.3. Let $n = 3$, $d = 1$ and suppose that $\mathcal{P}$, regarded as a set of points of $PG(V) = PG(2, q)$, is a maximal arc $\mathcal{K}$ of given degree. Then $\mathrm{Exp}(\varepsilon)$ is the partial geometry $T_2^*(\mathcal{K})$. In particular, when q is even and $\mathcal{K}$ is a hyperoval O, we get the generalized quadrangle $T_2^*(O)$.

Example 2.4. Again, $n = 3$ and $d = 1$, but now we assume that $\mathcal{P}$ forms an oval O in $PG(2,q)$. Then $\mathrm{Exp}(\varepsilon)$ is the subgeometry of the generalized quadrangle $T_2(O)$ far from the distinguished point ∞ of $T_2(O)$ (notation as in Payne and Thas [29, Chapter 3]). Similarly, if $d = 1$ but $n = 4$ and $\mathcal{P}$ forms an ovoid O in $PG(3,q)$, then $\mathrm{Exp}(\varepsilon)$ is the subgeometry of $T_3(O)$ far from ∞.

Example 2.5. Let $n = 3$, $d = 2$ and suppose that $\mathcal{P}$ is a dual oval of $PG(2,q)$. Then $\mathrm{Exp}(\varepsilon)$ is the dual of a Laguerre plane. When q is even, we may also consider the case where $\mathcal{P}$ forms a dual hyperoval of $PG(2,q)$. In that case, $\mathrm{Exp}(\varepsilon)$ is the dual of a special Laguerre plane.

Example 2.6. Let $q = 2$ and suppose that $\mathcal{P}$ is a $(d-1)$-dimensional dual hyperoval of $PG(n-1,2)$ (Del Fra [13], Yoshiara [40]). Then $\mathrm{Exp}(\varepsilon)$ is a semibiplane.

3 Hulls

3.1 The abstract hull of an embedding

Let $\varepsilon : \Gamma \to G$ be an embedding. Denoting by $U(\varepsilon)$ the universal completion of the amalgam $\mathcal{A}(\varepsilon) := \{\varepsilon(X)\}_{X\in\Gamma}$ (see Ivanov [17, 1.3]), the composition $\tilde{\varepsilon}$ of ε with the natural embeddings of the groups $\varepsilon(X)$ in $U(\varepsilon)$ is an embedding of Γ in $U(\varepsilon)$ and the canonical projection of $U(\varepsilon)$ onto G induces a morphism $\pi_\varepsilon : \tilde{\varepsilon} \to \varepsilon$. By well known properties of universal completions one easily obtains the following:

Proposition 3.1. *For every embedding η of Γ and every morphism $f : \eta \to \varepsilon$, there exists a unique morphism $g : \tilde{\varepsilon} \to \eta$ such that $fg = \pi_\varepsilon$.* ■

In view of the above, we call $\tilde{\varepsilon}$ the *abstract hull* of ε (also the *hull* of ε, for short), and π_ε the *canonical projection* of $\tilde{\varepsilon}$ onto ε. Clearly, $\tilde{\varepsilon}$ is its own hull. The uniqueness claim of Proposition 3.1 implies the following:

Proposition 3.2. *All morphisms from $\tilde{\varepsilon}$ to itself are automorphism and all morphisms from $\tilde{\varepsilon}$ to ε are compositions of π_ε with automorphisms of $\tilde{\varepsilon}$.* ■

Theorem 3.3. *The geometry $\mathrm{Exp}(\tilde{\varepsilon})$ is the universal cover of $\mathrm{Exp}(\varepsilon)$ and, regarded $U(\varepsilon)$ and G as subgroups of $\mathrm{Aut}(\mathrm{Exp}(\tilde{\varepsilon}))$ and $\mathrm{Aut}(\mathrm{Exp}(\varepsilon))$, the group $U(\varepsilon)$ is the lifting of G to $\mathrm{Exp}(\tilde{\varepsilon})$.*

Proof. Let $\widetilde{\Delta}$ be the universal cover of $\Delta := \mathrm{Exp}(\varepsilon)$. Given a covering $\varphi : \widetilde{\Delta} \to \Delta$, the group $D(\varphi)$ of deck transformations of φ acts regularly on each of the fibers of φ. Moreover G, regarded as a subgroup of $\mathrm{Aut}(\Delta)$, lifts to a subgroup $\widetilde{G}$ of $\mathrm{Aut}(\widetilde{\Delta})$ containing $D(\varphi)$ as a normal subgroup and such that $G = \widetilde{G}/D(\varphi)$ (see [26, Theorem 12.13]). So, (2) of Proposition 2.8 holds. Consequently, $\widetilde{\Delta} \cong \mathrm{Exp}(\hat{\varepsilon})$ for a suitable embedding $\hat{\varepsilon} : \Gamma \to \widetilde{G}$ and ε is an image of $\hat{\varepsilon}$. By the universal properties of $\tilde{\varepsilon}$, there exists a morphism $\hat{f} : \tilde{\varepsilon} \to \hat{\varepsilon}$. Hence we also have a covering $\mathrm{Exp}(\hat{f}) : \mathrm{Exp}(\tilde{\varepsilon}) \to \mathrm{Exp}(\hat{\varepsilon})$ (Proposition 2.7). However, $\mathrm{Exp}(\hat{\varepsilon}) \cong \widetilde{\Delta}$ is simply connected. Therefore $\mathrm{Exp}(\hat{f})$ is an isomorphism and $\mathrm{Exp}(\tilde{\varepsilon}) \cong \widetilde{\Delta}$. By the second claim of Proposition 2.7 then $\hat{f}$ is an isomorphism and so $\tilde{\varepsilon} \cong \hat{\varepsilon}$. ■

Borrowing a word from Tits [35, Chapter 8], we say that an embedding is *abstractly dominant* (also *dominant*, for short) if it is its own hull.

Corollary 3.4. *An embedding ε is dominant if and only if its expansion* $\mathrm{Exp}(\varepsilon)$ *is simply connected.*

Example 3.1. Let $\Gamma = PG(V)$ for some $(n+1)$-dimensional K-vector space V and ε be the identity embedding, sending every proper non-trivial subspace of V to itself. Then $\mathrm{Exp}(\varepsilon) = AG(n+1, K)$. As affine geometries are simply connected, the additive group of V is the universal completion of the amalgam of proper linear subspaces of V, regarded as additive groups.

Example 3.2. For an integer $n > 1$, let Γ be an n-dimensional simplex with vertices $p_0, p_1, ..., p_n$ and let G be a commutative group splitting as a direct sum $G = \oplus_{i=0}^{n} G_i$ of non-trivial subgroups $G_0, G_1, ..., G_n$. For every proper nonempty subset J of $I := \{0, 1, ..., n\}$, put $G_J := \oplus_{j \in J} G_j$ and $\varepsilon(\{p_j\}_{j \in J}) := G_J$. Thus we get an abelian embedding $\varepsilon : \Gamma \to G$. As any two subgroups G_i, G_j commute as subgroups of $G_{i,j}$, G is the universal completion of the amalgam $\{G_J\}_{\emptyset \neq J \subset I}$. So, $G = U(\varepsilon)$, ε is abstractly dominant and $\mathrm{Exp}(\varepsilon)$ is simply connected. In fact, $\mathrm{Exp}(\varepsilon)$ is 2-simply connected; indeed, it is a non-thick building of type C_{n+1}:

Example 3.3. Let Γ be the Coxeter complex of type C_n, regarded as the system of cliques of a complete n-partite graph with $2n$ vertices $p_{1,0}, ..., p_{n,0}, p_{1,1}, ..., p_{n,1}$ and n classes $\{p_{i,0}, p_{i,1}\}$ of size 2. Given a basis $B := \{b_{i,0}\}_{i=1}^{n} \cup \{b_{i,1}\}_{i=1}^{n}$ of $V := V(2n, 2)$, for every proper nonempty subset J of $\{1, ..., n\}$ and every mapping $f : J \to \{0, 1\}$ put $\varepsilon(\{p_{j,f(j)}\}_{j \in J}) := \langle b_{j,f(j)} \rangle_{j \in J}$. Thus, we get a lax projective embedding $\varepsilon : \Gamma \to PG(V)$. The expansion $\mathrm{Exp}(\varepsilon)$ is a thin geometry belonging to the affine diagram $\widetilde{C}_n$:

The residues of the maximal elements of $\mathrm{Exp}(\varepsilon)$ are Coxeter complexes of type C_n, obtained as affine expansions of the embeddings induced by ε on maximal elements of Γ (compare Example 3.2). Hence the universal cover of $\mathrm{Exp}(\varepsilon)$ is a Coxeter complex and, by Theorem 3.3, $U(\varepsilon)$ can be recovered as a subgroup of the Coxeter group of type $\widetilde{C}_n$. Explicitly, given an orthonormal basis $\{u_i\}_{i=1}^{n}$ of the n-dimensional real vector space $V(n, \mathcal{R})$ for $i = 1, 2, ..., n$ let τ_i be the translation of $AG(n, \mathcal{R})$ sending every vector $x \in V(n, \mathcal{R})$ to $x + u_i$ and let r_i be the orthogonal reflexion of $V(n, \mathcal{R})$ with the hyperplane $u_i^{\perp}$ as the axis. Then, regarded the points of $\mathrm{Exp}(\varepsilon)$ as vectors of $V(n, \mathcal{R})$ with integral coordinates, $U(\varepsilon) = \langle \tau_i r_i, \tau_i^{-1} r_i \rangle_{i=1}^{n}$.

Example 3.4. Suppose that Γ has rank 1. Then $U(\varepsilon)$ is the free product of the groups $\varepsilon(p)$ $(p \in P(\Gamma))$ and $\mathrm{Exp}(\tilde{\varepsilon})$ is a building of affine type $\widetilde{A}_1$.

3.2 Abelian and linear hulls

Given an abelian embedding $\varepsilon : \Gamma \to G$, the quotient $U_{\mathrm{ab}}(\varepsilon) := U(\varepsilon)/U(\varepsilon)'$ of $U(\varepsilon)$ by its commutator subgroup $U(\varepsilon)'$ is the universal abelian completion of $\mathcal{A}(\varepsilon) :=$

$\{\varepsilon(X)\}_{X\in\Gamma}$. The composition $\tilde{\varepsilon}_{\rm ab}$ of ε with the natural embeddings of the groups $\varepsilon(X)$ in $U_{\rm ab}(\varepsilon)$ is an embedding of Γ in $U_{\rm ab}(\varepsilon)$. Properties similar to those stated in Propositions 3.1 and 3.2 hold for $\tilde{\varepsilon}_{\rm ab}$, too (but now in the category of abelian embeddings). We call $\tilde{\varepsilon}_{\rm ab}$ the *abelian hull* of ε. When $\tilde{\varepsilon}_{\rm ab} = \varepsilon$ we say that ε is *commutatively dominant*.

Suppose furthermore that ε is K-linear for a given division ring K and let $V = \mathrm{cod}(\varepsilon)$. Then all groups $\varepsilon(X) \in \mathcal{A}(\varepsilon)$ are equipped with a structure of a K-vector space. We inductively define a sequence $A_0(K), A_1(K), A_2(K), \ldots$ of subgroups of $U_{\rm ab}(\varepsilon)$, as follows: $A_0(K) := 0$ and $A_{n+1}(K)$ is the subgroup of $U_{\rm ab}(\varepsilon)$ generated by the sums $\sum_{i=1}^m x_i$ (additive notation, as $U_{\rm ab}(\varepsilon)$ is abelian) where $x_1 \in \varepsilon(p_1)$, $x_2 \in \varepsilon(p_2)$,..., $x_m \in \varepsilon(p_m)$, $p_1, p_2, \ldots, p_m \in P(\Gamma)$ and $\sum_{i=1}^m kx_i \in A_n(K)$ for some non-zero scalar $k \in K^*$, the products kx_i being computed in the vector spaces $\varepsilon(p_i)$, the latters being regarded as subgroups of $U_{\rm ab}(\varepsilon)$ (as we can do, since $\varepsilon(X) \cap U(\varepsilon)' = 1$ for every $X \in \Gamma$). We put $A(K) := \cup_{n=0}^{\infty} A_n(K)$.

Lemma 3.5. *We have $\varepsilon(X) \cap A(K) = 0$ for every $X \in \Gamma$. Furthermore, for every scalar $k \in K$ and any choice of points $p_1, \ldots, p_m$ of Γ and elements $x_1 \in \varepsilon(p_1), \ldots, x_m \in \varepsilon(p_m)$, if $\sum_{i=1}^m x_i \in A(K)$ then $\sum_{i=1}^m kx_i \in A(K)$.*

Proof. $A(K)$ is contained in the kernel of the canonical projection of $U_{\rm ab}(\varepsilon)$ onto the additive group of V. This remark proves the first claim of the Lemma. Turning to the second claim, let $\sum_{i=1}^m x_i \in A(K)$. Put $y_i := kx_i$ for $i = 1, 2, \ldots, m$ and $y := \sum_{i=1}^m y_i$. If $y = 0$ there is nothing to prove. Suppose $y \neq 0$. Hence $k \neq 0$. As $\sum_{i=1}^m x_i \in A(K)$, we have $\sum_{i=1}^m x_i \in A_n(K)$ for some n. Therefore, $\sum_{i=1}^m k^{-1} y_i = \sum_{i=1}^m x_i \in A_n(K)$. Hence $\sum_{i=1}^m y_i \in A_{n+1}(K)$. ∎

By the second claim of Lemma 3.5, a K-vector space structure can be defined on the factor group $U_{\rm ab}(\varepsilon)/A(K)$ by putting

$$k \cdot (\sum_{i=1}^m x_i + A(K)) := \sum_{i=1}^m kx_i + A(K)$$

for $k \in K$, $p_1, p_2, \ldots, p_m \in P(\Gamma)$ and elements $x_1 \in \varepsilon(p_1)$, $x_2 \in \varepsilon(p_2)$,..., $x_m \in \varepsilon(p_m)$. We shall denote that vector space by $U_{\rm lin}(\varepsilon)$. By the first claim of Lemma 3.5, the composition of $\tilde{\varepsilon}_{\rm ab}$ with the projection of $U_{\rm ab}(\varepsilon)$ onto $U_{\rm lin}(\varepsilon) = U_{\rm ab}(\varepsilon)/A(K)$ is an embedding. Clearly, that embedding is K-linear and it is universal in the category of K-linear embeddings of Γ and linear morphisms. We denote it by $\tilde{\varepsilon}_{\rm lin}$ and we call it the *linear hull* of ε. If $\varepsilon = \tilde{\varepsilon}_{\rm lin}$, then we say that ε is *linearly dominant*.

Clearly, if ε is projective, then $\tilde{\varepsilon}_{\rm lin}$ is also projective. In particular, when Γ has rank 2 and ε is a full projective embedding, then $\tilde{\varepsilon}_{\rm lin}$ is just the hull of ε as constructed by Ronan [30, Proposition 3].

Proposition 3.6. *If K is a finite prime field, then $U_{lin}(\varepsilon) = U_{ab}(\varepsilon)$.*

Proof. Given $p_1, p_2, \ldots, p_m \in P(\Gamma)$ and $x_1 \in \varepsilon(p_1), x_2 \in \varepsilon(p_2), \ldots x_m \in \varepsilon(p_m)$ with $\sum_{i=1}^m x_i \neq 0$, the scalars $k \in K$ such that $\sum_{i=1}^m kx_i = 0$ form a subgroup $K(x_1, x_2, \ldots, x_m)$ of the additive group of K, but $1 \notin K(x_1, x_2, \ldots, x_m)$. On the other hand, the trivial subgroup is the only proper subgroup of the additive group of a finite prime field. Therefore, when K is finite and has prime order, $K(x_1, x_2, \ldots, x_m) = 0$. In that case, $A_1(K) = 0$, hence $A(K) = 0$. ∎

Note 3.1. Linearly dominant projective embeddings are often called *relatively universal* in the literature (see Shult [33], for instance).

3.3 Universal embeddings

We say that two embeddings ε and η of a given geometry Γ are *locally isomorphic*, and we write $\varepsilon \sim \eta$, if the amalgams $\mathcal{A}(\varepsilon)$ and $\mathcal{A}(\eta)$ are isomorphic, namely: there exists a family $\alpha = \{\alpha_X\}_{X\in\Gamma}$ of isomorphisms $\alpha_X : \varepsilon(X) \stackrel{\cong}{\to} \eta(X)$ such that, if $X \leq Y$, then α_X is the restriction of α_Y to $\varepsilon(X)$.

Clearly, every isomorphism from ε to η is a local isomorphism. Every isomorphism from the abstract hull $\tilde{\varepsilon}$ of ε to the abstract hull $\tilde{\eta}$ of η also induces a local isomorphism from ε to η. Conversely, if $\alpha = \{\alpha_X\}_{X\in\Gamma}$ is a local isomorphism from ε to η, then α lifts to an isomorphism $\tilde{\alpha}$ from the universal completion $U(\varepsilon)$ of the amalgam $\mathcal{A}(\varepsilon)$ to the universal completion $U(\eta)$ of $\mathcal{A}(\eta)$. For every $X \in \Gamma$, $\tilde{\alpha}$ induces α_X from $\varepsilon(X)$ ($\leq U(\varepsilon)$) to $\eta(X)$ ($\leq U(\eta)$). Hence $\tilde{\alpha}$ is an isomorphism from $\tilde{\varepsilon}$ to $\tilde{\eta}$.

Assume furthermore that both ε and η are abelian. Then $\tilde{\alpha}$ maps the commutator subgroup of $U(\varepsilon)$ onto the commutator subgroup of $U(\eta)$, whence it induces an isomorphism $\tilde{\alpha}_{\mathrm{ab}}$ from the abelian hull $\tilde{\varepsilon}_{\mathrm{ab}}$ of ε to the abelian hull $\tilde{\eta}_{\mathrm{ab}}$ of η. We summarize the above as follows:

Theorem 3.7. *For two embeddings ε and η of Γ, we have $\varepsilon \sim \eta$ if and only if $\tilde{\varepsilon} \cong \tilde{\eta}$. Furthermore, if both ε and η are abelian, then $\varepsilon \sim \eta$ if and only if $\tilde{\varepsilon}_{ab} \cong \tilde{\eta}_{ab}$.* ∎

It is clear from the proof of Proposition 2.1 that the relation $\sim$ admits infinitely many classes; that is, every geometry admits infinitely many embeddings with pairwise non-isomorphic hulls. However, assume we are interested in a particular category $\mathcal{C}$ of embeddings of a given geometry Γ. By Theorem 3.7, $\mathrm{Obj}(\mathcal{C})$ is contained in a class of $\sim$ if and only if all objects of $\mathcal{C}$ have the same hull, which we call the *hull* of $\mathcal{C}$.

Suppose that $\mathrm{Obj}(\mathcal{C})$ is contained in a class of $\sim$ and let $\tilde{\varepsilon}$ be the hull of $\mathcal{C}$. Suppose furthermore that $\tilde{\varepsilon}$ admits an image $\tilde{\varepsilon}_{\mathcal{C}}$ belonging to $\mathcal{C}$ and satisfying 'universal' properties like those of Propositions 3.1 and 3.2, but referred to objects and morphisms of $\mathcal{C}$. Then we call $\tilde{\varepsilon}_{\mathcal{C}}$ the *$\mathcal{C}$-universal embedding* of Γ. Clearly, if $\tilde{\varepsilon} \in \mathrm{Obj}(\mathcal{C})$, then $\tilde{\varepsilon} = \tilde{\varepsilon}_{\mathcal{C}}$.

Still assuming that $\mathrm{Obj}(\mathcal{C})$ is contained in a class of $\sim$, suppose that all embeddings $\varepsilon \in \mathrm{Obj}(\mathcal{C})$ are abelian. Then, in view of Theorem 3.7, all objects of $\mathcal{C}$ have the same abelian hull $\tilde{\varepsilon}_{\mathrm{ab}}$, which we call the *abelian hull* of $\mathcal{C}$. If $\tilde{\varepsilon}_{\mathrm{ab}} \in \mathrm{Obj}(\mathcal{C})$, then $\tilde{\varepsilon}_{\mathrm{ab}}$ is the $\mathcal{C}$-universal embedding.

Turning to K-linear embeddings for a given division ring K, we consider the following refinement of the relations $\sim$: Given two K-linear embeddings ε and η of Γ, we say that an isomorphism $\alpha = \{\alpha_X\}_{X\in\Gamma}$ from $\mathcal{A}(\varepsilon)$ to $\mathcal{A}(\eta)$ is *linear* if $\alpha_X : \varepsilon(X) \stackrel{\cong}{\to} \eta(X)$ is a semilinear isomorphism of K-vector spaces, for every $X \in \Gamma$. We say that ε and η are *locally linearly isomorphic* and we write $\varepsilon \sim_{\mathrm{lin}} \eta$ if $\mathcal{A}(\varepsilon)$ and $\mathcal{A}(\eta)$ are linearly isomorphic.

Theorem 3.8. *For two K-linear embeddings ε and η of Γ, we have $\varepsilon \sim_{lin} \eta$ if and only if $\tilde{\varepsilon}_{lin} \cong_{lin} \tilde{\eta}_{lin}$.*

Proof. We only sketch a proof of the 'only if' claim. Given a linear isomorphism α from $\mathcal{A}(\varepsilon)$ to $\mathcal{A}(\eta)$, we consider its lifting $\tilde{\alpha}_{\mathrm{ab}} : U_{\mathrm{ab}}(\varepsilon) \overset{\cong}{\to} U_{\mathrm{ab}}(\eta)$. For $\theta = \varepsilon$ or η, we consider the subgroups $A_n(K)$ and $A(K)$ of $U_{\mathrm{ab}}(\theta)$ defined in Subsection 3.2, but now we denote them by $A_n^\theta(K)$ and $A^\theta(K)$. By induction on n, one can prove that $\tilde{\alpha}_{\mathrm{ab}}(A_n^\varepsilon(K)) = A_n^\eta(K)$ for every n. Hence $\tilde{\alpha}_{\mathrm{ab}}(A^\varepsilon(K)) = A^\eta(K)$. Consequently, $\tilde{\alpha}_{\mathrm{ab}}$ induces an isomorphism $\tilde{\alpha}_{\mathrm{lin}}$ from $U_{\mathrm{lin}}(\varepsilon) = U_{\mathrm{ab}}(\varepsilon)/A^\varepsilon(K)$ to $U_{\mathrm{lin}}(\eta) = U_{\mathrm{ab}}(\eta)/A^\eta(K)$. Regarding $U_{\mathrm{lin}}(\varepsilon)$ and $U_{\mathrm{lin}}(\eta)$ as K-vector spaces, $\tilde{\alpha}_{\mathrm{lin}}$ is a semilinear mapping. ■

By Theorem 3.8, if all embeddings $\varepsilon \in \mathrm{Obj}(\mathcal{C})$ are K-linear and $\mathrm{Obj}(\mathcal{C})$ is contained in a class of $\sim_{\mathrm{lin}}$, then all objects of $\mathcal{C}$ have the same linear hull $\tilde{\varepsilon}_{\mathrm{lin}}$. We call it the *linear hull* of $\mathcal{C}$. Clearly, if $\tilde{\varepsilon}_{\mathrm{lin}} \in \mathrm{Obj}(\mathcal{C})$ and all morphism of $\mathcal{C}$ are linear, then $\tilde{\varepsilon}_{\mathrm{lin}}$ is the $\mathcal{C}$-universal embedding of Γ.

Universal representation groups. Suppose that Γ has rank 2, that all lines of Γ have exactly three points and no two of them have more than one point in common. Let $\mathcal{R}$ be the class of all locally $GF(2)$-projective embeddings of Γ, namely the embeddings $\varepsilon : \Gamma \to G$ such that:

(R1) $\varepsilon(p)$ has order 2 for every point $p \in P(\Gamma)$ and,

(R2) $\varepsilon(L)$ is elementary abelian of order 4, for every line $L \in L(\Gamma)$.

Suppose that $\mathcal{R} \neq \emptyset$, namely Γ admits an embedding ε satisfying (R1) and (R2). Clearly, all members of $\mathcal{R}$ are mutually locally isomorphic and the abstract hull $\tilde{\varepsilon}$ of ε belongs to $\mathcal{R}$. So, regarded $\mathcal{R}$ as a full subcategory of the category of embeddings of Γ, $\tilde{\varepsilon}$ is the $\mathcal{R}$-universal embedding of Γ. It is called the *universal representation* of Γ and its codomain is the *universal representation group* of Γ (Ivanov and Shpectorov [18]).

Suppose furthermore that $\mathcal{R}$ contains an abelian member ε and let $\mathcal{R}_{\mathrm{ab}}$ be the subclass of $\mathcal{R}$ formed by the abelian members of $\mathcal{R}$. The abelian hull $\tilde{\varepsilon}_{\mathrm{ab}}$ of ε belongs to $\mathcal{R}_{\mathrm{ab}}$, hence it is $\mathcal{R}_{\mathrm{ab}}$-universal. We call it the *universal abelian representation* of Γ. Its codomain is called the *universal representation module* of Γ (Ivanov and Shpectorov [18]). All members of $\mathcal{R}_{\mathrm{ab}}$ are full projective embeddings and $\tilde{\varepsilon}_{\mathrm{ab}}$ is the absolutely universal projective embedding of Γ (see the next paragraph).

Absolutely universal projective embeddings. Given a division ring K and a geometry Γ admitting a full K-projective embedding, let $\mathcal{P}_K$ be the category of full K-projective embeddings of Γ, with linear morphisms between them. Suppose that $\mathrm{Obj}(\mathcal{P}_K)$ is contained in a class of $\sim_{\mathrm{lin}}$. Then $\mathcal{P}_K$ admits the linear hull $\tilde{\varepsilon}_{\mathrm{lin}}$ and $\tilde{\varepsilon}_{\mathrm{lin}} \in \mathrm{Obj}(\mathcal{P}_K)$. So, $\tilde{\varepsilon}_{\mathrm{lin}}$ is $\mathcal{P}_K$-universal. The embedding $\tilde{\varepsilon}_{\mathrm{lin}}$ is called the *absolutely universal* K-projective embedding of Γ (Shult [33]); also, the *universal* or the *absolute* K-projective embedding of Γ, for short (Kasikova and Shult [20]). So,

Proposition 3.9. *A geometry Γ admits the absolutely universal K-projective embedding if and only if all full K-projective embeddings of Γ belong to the same class of $\sim_{lin}$.* ■

A similar result can be stated for the category of lax K-projective embeddings.

4 Hulls and extensions

In this section Γ has rank at least 3, $\varepsilon : \Gamma \to G$ is an embedding and ε_0 is the embedding induced by ε on the point-line system $\Sigma := (P(\Gamma), L(\Gamma))$ of Γ. Following the notation of Section 3, we denote by $\tilde{\varepsilon}$ and $\tilde{\varepsilon}_0$ the hulls of ε and ε_0 and, for every $X \in \Gamma$ we put

$$\hat{\varepsilon}(X) := \langle \tilde{\varepsilon}_0(p) \rangle_{p \in P(X)} \ (\leq U(\varepsilon_0)).$$

Thus, we obtain a mapping $\hat{\varepsilon}$ from Γ to the subgroup lattice of $U(\varepsilon_0)$ (compare Subsection 2.5). As we shall prove in a few lines, $\tilde{\varepsilon}_0$ extends to Γ and $\hat{\varepsilon}$ is its extension. As the embedding induced by $\tilde{\varepsilon}$ on Σ dominates ε_0, that induced embedding is an image of $\tilde{\varepsilon}_0$. Therefore, $U(\varepsilon)$ is a homomorphic image of $U(\varepsilon_0)$. We denote by π_0 the canonical projection of $U(\varepsilon_0)$ onto $U(\varepsilon)$. Clearly, $\pi_0(\hat{\varepsilon}(X)) = \tilde{\varepsilon}(X)$ for every $X \in \Gamma$, but the restriction of π_0 to $\hat{\varepsilon}(X)$ might be non-injective when $t(X) > 1$.

Theorem 4.1. *All the following hold:*

(1) $\tilde{\varepsilon}_0$ *extends to* Γ *and* $\hat{\varepsilon}$ *is the extension of* $\tilde{\varepsilon}_0$ *to* Γ*;*

(2) *the embedding* $\hat{\varepsilon}$ *is dominant;*

(3) π_0 *induces a* 2*-covering* $\mathrm{Exp}(\pi_0) : \mathrm{Exp}(\hat{\varepsilon}) \to \mathrm{Exp}(\tilde{\varepsilon})$*, which sends* g *to* $\pi_0(g)$ *and* $g\hat{\varepsilon}(X)$ *to* $\pi_0(g)\cdot\tilde{\varepsilon}(X)$*, for* $g \in U(\varepsilon_0)$ *and* $X \in \Gamma$*.*

Proof. As π_0 maps $\hat{\varepsilon}(X)$ onto $\tilde{\varepsilon}(X)$ and $\tilde{\varepsilon} : \Gamma \to U(\varepsilon)$ is an embedding, $\hat{\varepsilon}$ satisfies (E1), whence it is the extension of $\tilde{\varepsilon}_0$. Claim (1) is proved. As $U(\varepsilon_0)$ is the codomain of $\hat{\varepsilon}$, it is a homomorphic image of $U(\hat{\varepsilon})$. On the other hand, $U(\hat{\varepsilon})$ is a homomorphic image of $U(\varepsilon_0)$, as it contains a copy of the amalgam $\{\varepsilon_0(X)\}_{X\in\Sigma}$ of which $U(\varepsilon_0)$ is the universal completion. Therefore, $U(\hat{\varepsilon}) = U(\varepsilon_0)$. Hence $\hat{\varepsilon}$ is dominant, as claimed in (2). Claim (3) follows from the fact that $\mathrm{Ker}(\pi_0) \cap \hat{\varepsilon}(L) = 1$ for every line $L \in L(\Gamma)$. ■

Corollary 4.2. *Both the following hold:*

(1) *If* ε_0 *is dominant, then* $\varepsilon = \tilde{\varepsilon} = \hat{\varepsilon}$*.*

(2) $\mathrm{Exp}(\hat{\varepsilon})$ *is simply connected and is a* 2*-cover of* $\mathrm{Exp}(\varepsilon)$*.*

Proof. (1) follows from Theorem 4.1(2). The first claim of (2) follows from Theorem 4.1(2) and Corollary 3.4. The second claim of (2) follows from Theorem 4.1(3) and the fact that $\mathrm{Exp}(\tilde{\varepsilon})$ is a cover of $\mathrm{Exp}(\varepsilon)$. ■

When ε is abelian (in particular, linear), results similar to the above hold for the abelian (linear) hull of ε_0, but we are not going to prove them here. Instead, we make some comments on Theorem 4.1.

When π_0 is not injective, $\mathrm{Exp}(\hat{\varepsilon})$ is a proper 2-cover of $\mathrm{Exp}(\tilde{\varepsilon})$. In this case $\mathrm{Exp}(\tilde{\varepsilon})$ is not 2-simply connected, although it is simply connected. On the other hand, $\mathrm{Exp}(\hat{\varepsilon})$ is simply connected by Corollary 3.4 and (2) of Theorem 4.1, but it might not be 2-simply connected. So, in general, we cannot claim that $\mathrm{Exp}(\hat{\varepsilon})$, which is a 2-cover of $\mathrm{Exp}(\varepsilon)$, is the universal 2-cover of $\mathrm{Exp}(\varepsilon)$. We will give a counterexample later (Example 4.1).

Theorem 4.3. *Suppose that* Γ *is* 2*-simply connected. Then* $\mathrm{Exp}(\hat{\varepsilon})$ *is the universal* 2*-cover of* $\mathrm{Exp}(\varepsilon)$*.*

Proof. Let $\varphi : \Delta \to \mathrm{Exp}(\hat{\varepsilon})$ be the universal 2-covering of $\mathrm{Exp}(\hat{\varepsilon})$ and $\mathrm{Tr}(\Delta)$ be the $\{0,1,2\}$-truncation of Δ. As Γ is 2-simply connected and the point-residues of $\mathrm{Exp}(\hat{\varepsilon})$ are isomorphic to Γ, φ induces isomorphisms between the point-residues of Δ and the corresponding point-residues of $\mathrm{Exp}(\hat{\varepsilon})$. Hence it induces a covering $\mathrm{Tr}(\varphi)$ from $\mathrm{Tr}(\Delta)$ to the $\{0,1,2\}$-truncation of $\mathrm{Exp}(\hat{\varepsilon})$. The latter is equal to $\mathrm{Exp}(\tilde{\varepsilon}_0)$, which is simply connected by Theorem 3.3. Hence $\mathrm{Tr}(\varphi)$ is an isomorphism.

Suppose $P(X) \subseteq P(Y)$ for two elements $X, Y \in \Delta$. Then $P(\varphi(X)) \subseteq P(\varphi(Y))$, hence $\varphi(X) \leq \varphi(Y)$ by the Point-Set Property (PS) in $\mathrm{Exp}(\hat{\varepsilon})$. However, given $p \in P(X)$, φ induces an isomorphism from $\mathrm{Res}_\Delta(p)$ to the residue of $\varphi(p)$ in $\mathrm{Exp}(\hat{\varepsilon})$. Therefore, $X \leq Y$, as $p < X, Y$ and $\varphi(X) \leq \varphi(Y)$. Thus, Δ inherits the Point-Set Property (PS) from $\mathrm{Exp}(\hat{\varepsilon})$. That property and the fact that $\mathrm{Tr}(\varphi)$ is an isomorphism force φ to be an isomorphism. So, $\mathrm{Exp}(\hat{\varepsilon})$ is 2-simply connected. As $\mathrm{Exp}(\hat{\varepsilon})$ is a 2-cover of $\mathrm{Exp}(\varepsilon)$, it is the universal 2-cover of $\mathrm{Exp}(\varepsilon)$. ■

Theorem 4.4. *For* $X \in \Gamma$ *of type* $t(X) \geq 1$*, let* ε_X *be the embedding induced by* ε *on* $\mathrm{Res}^-_\Gamma(X)$ *as in Proposition* 2.4*. Suppose that* $\mathrm{Exp}(\varepsilon_X)$ *is simply connected for every* $X \in \Gamma$ *of type* $t(X) > 1$*. Then* $\tilde{\varepsilon} = \hat{\varepsilon}$*.*

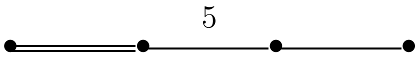
Proof. We only need to prove that $\mathrm{Ker}(\pi_0) \cap \hat{\varepsilon}(X) = 1$ for every $X \in \Gamma$. We shall argue by induction on $t(X)$. When $t(X) \leq 1$, there is nothing to prove. Suppose $t(X) > 1$. By the inductive hypothesis, $\mathrm{Ker}(\pi_0) \cap \hat{\varepsilon}(Y) = 1$ for every $Y < X$. Hence $\hat{\varepsilon}(X)$ is a homomorphic image of $U(\varepsilon_X)$. However, as $\mathrm{Exp}(\varepsilon_X)$ is simply connected by assumption, Corollary 3.4 implies that $U(\varepsilon_X) = \varepsilon(X)$. Hence $\mathrm{Ker}(\pi_0) \cap \hat{\varepsilon}(X) = 1$. ■

Corollary 4.5. *Suppose that the embedding* ε *is abelian* (*in particular, linear*) *and* $\mathrm{Exp}(\varepsilon_X)$ *is simply connected for every* $X \in \Gamma$ *of type* $t(X) > 1$*. Then the abelian* (*respectively, linear*) *hull of* ε *is the extension of the abelian* (*linear*) *hull of* ε_0*.*

Proof. By Theorem 4.4, $U(\varepsilon) = U(\varepsilon_0)$. Hence $U_{\mathrm{ab}}(\varepsilon) = U_{\mathrm{ab}}(\varepsilon_0) = U/U'$, where $U := U(\varepsilon) = U(\varepsilon_0)$. Similarly, in the linear case, $U_{\mathrm{lin}}(\varepsilon) = U_{\mathrm{lin}}(\varepsilon_0) = (U/U')/A(K)$. ■

Example 4.1. Let Γ be the quotient of the dodecahedron by the antipodality relation and $\varepsilon : \Gamma \to V$ be the 'free' $GF(2)$-linear embedding of Γ in $V := V(10,2)$, constructed as in the proof of Proposition 2.1. (Note that, as all lines of Γ have exactly two points, ε is laxly projective.) As any two points of Γ belong to some 2-element (namely, a pentagon), the amalgam $\{\varepsilon(X)\}_{X\in\Gamma}$ entails the information that $(\varepsilon(p_1), \varepsilon(p_2)) = 1$ for any two points $p_1, p_2 \in P(\Gamma)$. Hence $U(\varepsilon)$ is commutative. Consequently, $U(\varepsilon) = V$. That is, ε is dominant.

On the other hand, the point-line system Σ of Γ is isomorphic to the Petersen graph and the information $(\varepsilon(p_1), \varepsilon(p_2)) = 1$ is saved by the amalgam $\{\varepsilon(X)\}_{X\in\Sigma}$ only when p_1 and p_2 are collinear. So $U(\varepsilon_0)$ is the Coxeter group with Coxeter diagram isomorphic to the Petersen graph. Thus, $U(\varepsilon_0)$ is infinite. Hence $\mathrm{Exp}(\hat{\varepsilon})$ is a proper 2-cover of $\mathrm{Exp}(\hat{\varepsilon})$ ($= \mathrm{Exp}(\varepsilon)$). The geometry $\mathrm{Exp}(\hat{\varepsilon})$ is thin and belongs to the following non-spherical Coxeter diagram:



By a well-known theorem of Tits [36], the universal 2-cover of $\mathrm{Exp}(\hat{\varepsilon})$ is a Coxeter complex. However, $\mathrm{Exp}(\hat{\varepsilon})$ itself is not a Coxeter complex. Indeed, its point-residues are isomorphic to Γ, which is a proper quotient of a Coxeter complex. Hence $\mathrm{Exp}(\hat{\varepsilon})$ is not 2-simply connected.

5 Embeddings of projective geometries

5.1 Full and sharp embeddings

Throughout this subsection $\Gamma = PG(n, K_1)$ for a given division ring K_1 and an integer $n > 1$ and $\varepsilon : \Gamma \to G$ is a given embedding. We say that the embedding ε is *full* if $\varepsilon(L) = \cup_{p \in P(L)} \varepsilon(p)$ for every line $L \in L(\Gamma)$.

Lemma 5.1. *Suppose that ε is full. Then*

$$(*) \quad G = \cup_{p \in P(\Gamma)} \varepsilon(p) \quad \textit{and } \varepsilon(X) = \cup_{p \in P(X)} \varepsilon(p) \quad \textit{for every } X \in \Gamma.$$

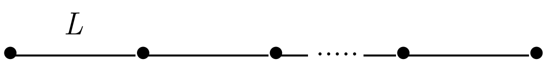
Proof. Let $g = g_1 g_2 ... g_m$ for $g_i \in \varepsilon(p_i)$, $p \in P(\Gamma)$ for $i = 1, 2, ..., m$. We shall prove, by induction on m, that $g \in \varepsilon(p)$ for some point p. When $m = 1$ there is nothing to prove. Let $m > 1$. As ε is full, we have $g_{m-1} g_m = g' \in \varepsilon(p')$ for a suitable point p' of the line through p_{m-1} and p_m. Then $g = g_1 .... g_{m-2} g'$ and the conclusion follows by the inductive hypothesis. ■

Theorem 5.2. *If ε is full, then it is dominant.*

Proof. If ε is full, then the hull $\tilde{\varepsilon}$ of ε is also full. Hence $(*)$ of Lemma 5.1 holds for $\tilde{\varepsilon}$. Accordingly, any two points of $\mathrm{Exp}(\tilde{\varepsilon})$ are collinear (compare the second claim of Proposition 2.11). Therefore, $\mathrm{Exp}(\tilde{\varepsilon})$ does not admit any proper quotient. By Theorem 3.3, $\mathrm{Exp}(\tilde{\varepsilon}) = \mathrm{Exp}(\varepsilon)$. Hence $\tilde{\varepsilon} = \varepsilon$. ■

If $\varepsilon(p_1) \cap \varepsilon(p_2) = 1$ for any two points $p_1, p_2 \in P(\Gamma)$, then we say that ε is *sharp*. Clearly, ε is full and sharp if and only if $\{\varepsilon(p)\}_{p \in P(L)}$ is a partition of the group $\varepsilon(L)$, for every line L. (See Subsection 2.8 for the definition of partitions of groups.) The next Lemma easily follows from the third claim of Proposition 2.11.

Lemma 5.3. *If ε is full and sharp, then* $\mathrm{Exp}(\varepsilon)$ *belongs to the following diagram:*



■

Theorem 5.4. *Suppose that K_1 is finite, ε is full and sharp and there exist an integer $k > 1$ such that $|\varepsilon(p)| = k$ for every point $p \in P(\Gamma)$. Then $k = q := |K_1|$ and G is the additive group of $V(n+1, q)$.*

Proof. By Lemma 5.3 and a well known theorem of Doyen and Hubaut [14], we have one of the following cases:

(1) $k = q$ and $\mathrm{Exp}(\varepsilon) \cong AG(n+1, q)$;

(2) $k = q + 1$ and $\mathrm{Exp}(\varepsilon) \cong PG(n+1, q)$;

(3) $n = 3$ and $q = k^2$.

In case (1), $\mathrm{Exp}(\varepsilon) \cong AG(n+1, q)$ and G acts on $\mathrm{Exp}(\varepsilon)$ as the translation group of $AG(n+1, q)$. In this case, we are done. Assume (2). Then, for every line $L \in L(\Gamma)$, we have $|\varepsilon(L)| = q^2 + q + 1$, since the plane $\varepsilon(L)$ of $\mathrm{Exp}(\varepsilon)$ has $q^2 + q + 1$ points and the points of that plane are the elements of $\varepsilon(L)$. Hence $\varepsilon(L)$ has no subgroup of order $q + 1$, contrary to the assumption that $k = q + 1$. So, case (2) is impossible.

Finally, let (3) hold. Then every plane of $\mathrm{Exp}(\varepsilon)$ has $(q+1)(k-1)+1 = (q+1)k-q$ points. Therefore, $|\varepsilon(L)| = (q + 1)k - q = k(k^2 - k + 1)$ for every line $L \in L(\Gamma)$. As $(k, k^2 - k + 1) = 1$, the group $\varepsilon(L)$ contains some elements of order coprime to k. However, as ε is full, the order of every element of $\varepsilon(L)$ is a divisor of k. Again, we have reached a contradiction. qed

The hypothesis that all subgroups $\varepsilon(p)$ have the same order is crucial for Theorem 5.4, as shown by the following example.

Example 5.1. Let K be the elementary abelian group of order 3^n for some $n > 1$ and ι be the automorphism of K inverting all elements of K. Put $H = \langle \iota \rangle$ and $G = KH$ (so, G is a Frobenius group with kernel K and complement H). Define a geometry Γ as follows: for every $i = 0, 1, ..., n-1$, the i-elements of Γ are the subgroups of K of order 3^{i+1} and all subgroups of G of order $3^i 2$. The incidence relation is the natural one, namely inclusion between subgroups. It is not difficult to check that $\Gamma \cong PG(n, 3)$.

By definition, Γ is a subposet of the poset $\mathcal{S}(G)$ of subgroups of G. The inclusion mapping $\varepsilon : \Gamma \hookrightarrow \mathcal{S}(G)$ is an embedding and $\{\varepsilon(p)\}_{p \in P(X)}$ is a partition of $\varepsilon(X)$, for every element $X \in \Gamma$ of positive type. However, the conclusions of Theorem 5.4 fail to hold. In fact, the groups $\varepsilon(p)$ ($p \in P(\Gamma)$) have different orders, namely 2 and 3.

Theorem 5.5. *For a division ring K, let ε be a K-linear embedding of $\Gamma = PG(n, K_1)$ in a K-vector space V. Suppose that ε is full and sharp. Then K_1 is an extension of K. Furthermore,*

$$(**) \quad \langle \varepsilon(p) \rangle_{p \in S} = \cup_{p \in S} \varepsilon(p) = \oplus_{p \in P} \varepsilon(p)$$

for every (possibly improper) subspace S of Γ and every independent spanning set P of S.

Proof. We shall firstly prove the following:

$$(1) \quad \langle \varepsilon(p) \rangle_{p \in S} = \oplus_{p \in P} \varepsilon(p)$$

for every subspace S of Γ and every independent spanning set P of S. By Lemma 5.1, for S and P as above we have

$$(2) \quad \langle \varepsilon(p) \rangle_{p \in S} = \cup_{p \in S} \varepsilon(p) = \langle \varepsilon(p) \rangle_{p \in P}.$$

Suppose $v_1 + v_2 + ... + v_m = 0$ for $v_1 \in \varepsilon(p_1)$, $v_2 \in \varepsilon(p_2)$,..., $v_m \in \varepsilon(p_m)$ and mutually distinct points $p_1, p_2, ..., p_m \in P$. However, $v_2 + ... + v_m \in \varepsilon(p)$ for some point p of the subspace S' spanned by $p_2, p_3, ..., p_m$, by (2) applied to S'. So, $v_1 = -v_2 - ... - v_m \in \varepsilon(p)$. Therefore, and since ε is sharp, either $v_1 = 0$ or $p_1 = p'$. However, $p_1 \notin S'$ since P is independent. Hence $v_1 = 0$. Similarly, $v_i = 0$ for every $i = 2, 3, ..., m$. So, (1) is proved.

Let now p_0, p_1, p_2 be three non-collinear points of Γ and S the plane of Γ spanned by them. For $i = 0, 1, 2$, pick a non-zero vector $v_i \in \varepsilon(p_i)$. By (1), the vectors v_0, v_1, v_2 are independent in $V := \mathrm{cod}(\varepsilon)$. Let S_0 be the plane of $PG(V)$ spanned by them. By Lemma 5.1 and since ε is K-linear, every point x of S_0 is contained in $\varepsilon(\eta(x))$ for a unique point $\eta(x)$ of Γ. The function η defined in this way sends collinear triples of points of S_0 onto collinear triples of points of Γ. So, η is an embedding of S_0 in the plane S of Γ spanned by p_0, p_1, p_2. As K coordinatizes S_0 whereas S is coordinatized by K_1, the latter is an extension of K. ■

Corollary 5.6. *With K, V and $\varepsilon : \Gamma = PG(n, K_1) \to V$ as in the hypotheses of Theorem 5.5, suppose furthermore that $\dim(\varepsilon(p)) = d$ for a given positive integer d and every point $p \in P(\Gamma)$. Then $|K_1 : K| = d$, $\dim(V) = (n+1)d$ and ε is induced by the natural inclusion of $V(n+1, K_1)$ in $V((n+1)d, K) \cong V$.*

Proof. The equality $\dim(V) = (n+1)d$ follows from $(**)$ of Theorem 5.5 applied to an independent spanning set P of $\Gamma = PG(n, K_1)$. Again by $(**)$ of Theorem 5.5, but applied to planes of Γ, and repeating the argument exploited in the second part of the proof of Theorem 5.5, we obtain the rest of the statement. ■

The hypothesis that $\dim(\varepsilon(p))$ does not depend on the choice of the point $p \in P(\Gamma)$ cannot be removed from Corollary 5.6, as shown by the following example.

Example 5.2. Given Γ, K, V, ε and d as in Corollary 5.6, let V_0 be a 1-dimensional linear subspace of V. Put $\eta(X) := (\varepsilon(X) + V_0)/V_0$, for every element $X \in \Gamma$. The mapping η defined in this way is a full, sharp K-linear embedding of Γ in V/V_0. However, $\dim(\eta(p)) = d$ or $d - 1$ according to whether $V_0 \cap \varepsilon(p) = 0$ or $V_0 \leq \varepsilon(p)$.

5.2 Lax locally projective embeddings

With $\Gamma = PG(n, K_1)$ as in the previous subsection, suppose now that the embedding $\varepsilon : \Gamma \to G$ is laxly locally K-projective for a given division ring K, possibly different from K_1. We do not assume ε to be full. We neither assume it is projective. If ε is laxly K-projective, namely G is the additive group of a K-vector space V and $\varepsilon(p)$ is a subspace of V for every $p \in P(\Gamma)$, then we say that ε is *globally* projective, to avoid any confusion.

Lemma 5.7. *The group G is abelian.*

Proof. As ε is locally projective, the groups $\varepsilon(p)$ and $\varepsilon(L)$ are abelian for every point p and every line L of Γ. Therefore, as the points of Γ are mutually collinear, the groups $\varepsilon(p)$ mutually commute. By (E3), G is abelian. ■

As G is abelian, we will use the additive notation for products of elements of G. We define $A(K)$ as in Subsection 3.2, but in G instead of $U_{\mathrm{ab}}(\varepsilon)$. Clearly, if ε is globally projective, then $A(K) = 0$. The converse also holds true:

Theorem 5.8. *Let $A(K) = 0$. Then ε is globally projective, its codomain is $(n+1)$-dimensional as a K-vector space and K is an extension of K_1. Furthermore, ε is abstractly dominant. Consequently,* $\mathrm{Exp}(\varepsilon)$ *is simply connected.*

Proof. In the sequel, for $p \in P(\Gamma)$ and $L \in L(\Gamma)$, $V(p)$ and $V(L)$ are as in (P1) and (P2) of Subsection 2.2. As $A(K) = 0$, a K-vector space structure V can be defined on G by putting $k \cdot (\sum_{i=1}^{m} v_i) := \sum_{i=1}^{m} kv_i$ for every $k \in K$ and every choice of points $p_1, p_2, ..., p_m$ and elements $v_1 \in \varepsilon(p_1)$, $v_2 \in \varepsilon(p_2)$,..., $v_m \in \varepsilon(p_m)$. (Compare Subsection 3.2; needless to say, the products kv_i are done in the vector spaces $V(p_i)$.) Accordingly, $V(p)$ and $V(L)$ are subspaces of V for every $p \in P(\Gamma)$ and every $L \in L(\Gamma)$. Hence ε is K-linear. Moreover, it is K-projective.

We shall now prove that, for every element $X \in \Gamma$, the subspace $\varepsilon(X) < V$ has dimension $t(X) + 1$. When $t(X) \leq 1$, the claim holds by definition of lax projective embedding. We go on by induction on $i = t(X)$. Suppose $i > 1$. Then, given an element $Y < X$ of type $i - 1$ and a point $p_0 \in P(X) \setminus P(Y)$, we have $P(X) = \cup_{p \in P(Y)} P([p, p_0])$, where $[p, p_0]$ stands for the line of Γ through p and p_0. Hence $\varepsilon(X) = \langle \varepsilon(Y) \cup \varepsilon(p_0) \rangle$. As $\dim(\varepsilon(Y)) = i$ (by the inductive hypothesis) and $\varepsilon(p_0) \not\leq \varepsilon(Y)$ (because $p_0 \not\leq Y$), we get $\dim(\varepsilon(X)) = i + 1$. By a similar argument, we obtain that $\dim(V) = n + 1$. So, the image $\varepsilon(\Gamma)$ of Γ by ε is an n-dimensional subgeometry of $PG(V)$ and K is an extension of the underlying division ring K_ε of $\varepsilon(\Gamma)$. However, $\Gamma \cong \varepsilon(\Gamma)$. Hence $K_\varepsilon = K_1$.

Finally we prove that ε is abstractly dominant. Given a basis $\{p_i\}_{i=0}^{n}$ of the projective space Γ, we have $V = \oplus_{i=0}^{n} V(p_i)$ and, denoted by $L_{i,j}$ the line of Γ through p_i and p_j, V is the universal completion of the subamalgam $\{V(p_i)\}_{i=0}^{n} \cup \{V(L_{i,j})\}_{i<j}$ of $\{\varepsilon(X)\}_{X \in \Gamma}$. Therefore, $U(\varepsilon) = V$, namely ε is dominant. By Corollary 3.4, $\mathrm{Exp}(\varepsilon)$ is simply connected. ■

Corollary 5.9. *Assume that K is finite of prime order. Then $K_1 = K$, the codomain V of ε is an $(n+1)$-dimensional K-vector space and ε is an isomorphism from Γ to $PG(V)$.*

Proof. We have $A(K) = 0$, by same argument used in the proof of Proposition 3.6. Hence the conclusions of Theorem 5.8 hold. In particular, V is an n-dimensional vector space and K is an extension of K_1. However, K is prime. Hence $K_1 = K$ and $\varepsilon(\Gamma) = PG(V)$. ■

Corollary 5.10. *Suppose that ε is full. Then the same conclusions as in Corollary* 5.9 *hold.*

Proof. We first prove that $A(K) = 0$. Clearly, $A(K) = 0$ if and only if $A_1(K) = 0$. The equality $A_1(K) = 0$ is equivalent to the following: For every non-zero scalar $k \in K^*$, for any points $p_1, p_2, ..., p_m$ and elements $v_1 \in \varepsilon(p_1)$, $v_2 \in \varepsilon(p_2)$,..., $v_m \in \varepsilon(p_m)$, if

$$(1) \quad \sum_{i=1}^{m} kv_i = 0 \ \text{ for some } k \neq 0,$$

then

$$(2) \quad \sum_{i=1}^{m} v_i = 0.$$

We shall prove the implication (1) $\Rightarrow$ (2) by induction on m. When $m = 1$ there is nothing to prove and when $m = 2$ the implication easily follows from (P1), (P2) and (P3). Let $m > 2$. As ε is full, the element $v := v_{m-1} + v_m$ belongs to $\varepsilon(p)$ for a point p of the line of Γ through p_{m-1} and p_m. By (P2) and (P3), $kv = kv_{m-1} + kv_m$. Hence (1) can we rewritten as follows:

$$(3) \quad kv_1 + ... + kv_{m-2} + kv = 0$$

By the inductive hypothesis on m, (3) implies $v_1 + ... + v_{m-2} + v = 0$, which is the same as (2). Thus, $A_1(K) = 0$. Consequently, $A(K) = 0$ and the conclusions of Theorem 5.8 hold. In particular, $V := \mathrm{cod}(\varepsilon)$ is an $(n+1)$-dimensional vector space and K is an extension of K_1. As ε is full, $K_1 = K$. Hence $\varepsilon(\Gamma) = PG(V)$. ∎

The hypothesis that $A(K) = 0$ cannot be dropped from Theorem 5.8, as the following example shows.

Example 5.3. Let $\Gamma = PG(2, 2)$, regarded as a subgeometry of $PG(2, 8)$. The points and the lines of Γ are certain 1- and 2-dimensional linear subspaces of $V := V(3, 8)$, but there exist 31 points of $PG(2, 8)$ that are not contained in any line of Γ. Let P be one of those points and v a nonzero vector of the 1-dimensional linear space P. Put $V_0 := \{0, v\}$ and $G := V/V_0$ (a quotient of additive groups). For every subspace X of V corresponding to an element X of Γ, put $\varepsilon(X) := (X \oplus V_0)/V_0$. Then ε is a lax locally $GF(8)$-projective embedding of Γ in G, but G is not a $GF(8)$-vector space. In fact, $A(K) = P/V_0 \neq 0$.

5.3 Extensions

Still with $\Gamma = PG(n, K_1)$, let $\varepsilon_0 : \Sigma \to G$ be an embedding of the point-line system $\Sigma := (P(\Gamma), L(\Gamma))$ of Γ.

Theorem 5.11. *If ε_0 is full and sharp, then it extends to Γ.*

Proof. Property $(*)$ of Lemma 5.1 holds for ε_0 and for the restriction $\varepsilon_{0,X}$ of ε_0 to the point-line system of $\mathrm{Res}(X)$, for every element $X \in \Gamma$ of type $t(X) > 1$. Given $p_1 \in P(\Gamma)$ and $X \in \Gamma$ of type $t(X) > 1$, suppose that $\varepsilon_0(p_1) \leq \langle \varepsilon_0(p) \rangle_{p \in P(X)}$. By $(*)$ of Lemma 5.1, $\varepsilon_0(p_1) \cap \varepsilon_0(p_2) \neq 1$ for some $p_2 \in P(X)$. However, ε_0 is also assumed to be sharp. Hence $p_1 = p_2$. Therefore, $p_1 < X$. Thus, we have proved that the function sending X to $\langle \varepsilon_0(p) \rangle_{p \in P(X)}$ satisfies (E1). That is, ε_0 extends to Γ. ∎

Both the hypotheses of Theorem 5.11, namely fullness and sharpness, are essential for ε_0 to be extensible to Γ, as shown by the following counterexamples.

Example 5.4. With $\Gamma = PG(3, 2)$, let $\eta : \Gamma \to PG(3, 16)$ be a lax projective embedding as in Theorem 5.8. Given a basis $\{p_i\}_{i=0}^{3}$ of $PG(3, 2)$, for $i = 0, 1, 2, 3$ pick a non-zero vector $b_i \in \eta(p_i)$ so that to form a basis $\{b_i\}_{i=0}^{3}$ of $V := V(4, 16)$. Let ω be a primitive element of $GF(16)$ over $GF(2)$, put $v := \sum_{i=0}^{3} \omega^i b_i$ and let V_0 be the span of v in V. It is not difficult to see that $\eta(X) \cap V_0 = 0$ for any plane X of Γ. Therefore, if we put $\varepsilon_0(p) = (\eta(p) + V_0)/V_0$ and $\varepsilon_0(L) = (\eta(p) + V_0)/V_0$ for $p \in P(\Gamma)$ and $L \in L(\Gamma)$, we get a sharp but non-full lax projective embedding ε_0 of $(P(\Gamma), L(\Gamma))$ in $PG(V/V_0)$. However, ε_0 does not extend to Γ. Indeed, $\varepsilon_0(p_1) \leq \langle \varepsilon_0(p)\rangle_{p\in P(X)}$ for every point p and every plane X of Γ.

Example 5.5. Let $\Gamma = PG(3, 16)$ and $M_4(2)$ be the vector space of (4×4)-matrices over $GF(2)$. A matrix $M \in M_4(2)$ can be regarded as a vector of $V(4, 16)$ in two ways, as the 4-tuple of either its rows or its columns, each row or column of M corresponding to an element of the extension $GF(16)$ of $GF(2)$. Accordingly, we get two full and sharp $GF(2)$-linear embeddings ε_1 and ε_2 of Γ in $M_4(2)$: in the embedding ε_1 (respectively, ε_2) the coordinates of a vector of $V(4, 16)$ correspond to the rows (columns) of a matrix of $M_4(2)$. Chosen a point $p_0 \in P(\Gamma)$, we put $\varepsilon_0(p) := (\varepsilon_1(p)+\varepsilon_2(p_0))/\varepsilon_2(p_0)$ for $p \in P(\Gamma)$ and $\varepsilon_0(L) := (\varepsilon_1(L)+\varepsilon_2(p_0))/\varepsilon_2(p_0)$ for $L \in L(\Gamma)$. Then ε_0 is a $GF(2)$-linear embedding of $(P(\Gamma), L(\Gamma))$ in the factor space $V = M_4(2)/\varepsilon_2(p_0)$. However, ε_0 does not extend to Γ. Indeed, let X be a plane of Γ such that $\varepsilon_1(X) \cap \varepsilon_2(p_0) = 0$. (It is not difficult to see that Γ admits some planes with that property.) Then $\langle \varepsilon_0(p)\rangle_{p\in P(X)} = V$, which could not be if ε_0 extended to Γ.

The embedding ε_0 is full but non-sharp. Indeed, given a non-zero matrix $M_0 \in \varepsilon_2(p_0)$, a point $p_1 \in P(\Gamma)$ such that $M_0 \notin \varepsilon_1(p_1)$ and a matrix $M_1 \in \varepsilon_1(p_1)\backslash\varepsilon_2(p_0)$, let p_2 be the point of Γ such that $M_0 + M_1 \in \varepsilon_1(p_2)$. Then $p_1 \neq p_2$ but $\varepsilon_0(p_1)\cap\varepsilon_0(p_2) \supseteq M_1 + \varepsilon_2(p_0)$.

5.4 An application to locally projective geometries

In this subsection Γ is a poset-geometry of rank $n > 2$ satisfying the Point-Set property (PS) and Γ is *locally projective*, namely $\mathrm{Res}(A)$ is a non-degenerate projective geometry for every $(n-1)$-element A of Γ. We assume that the point-line system $\Sigma = (P(\Gamma), L(\Gamma))$ of Γ admits an embedding $\varepsilon_0 : \Sigma \to G$ and we denote the hull of ε_0 by $\tilde{\varepsilon}_0$. For $X \in \Gamma$ we put

$$\varepsilon(X) := \langle \varepsilon_0(p)\rangle_{p\in P(X)} \leq G, \qquad \hat{\varepsilon}(X) := \langle \tilde{\varepsilon}_0(p)\rangle_{p\in P(X)} \leq U(\varepsilon_0)$$

(we recall that $U(\varepsilon_0)$ is the codomain of $\tilde{\varepsilon}_0$), but we do not assume that ε_0 extends do Γ. By Theorem 4.1, if ε_0 extends to Γ then $\tilde{\varepsilon}_0$ also extends to Γ and ε and $\hat{\varepsilon}$ are the extensions of ε_0 and $\tilde{\varepsilon}_0$. As in Subsection 5.1, we say that ε_0 is *full* if $\cup_{p\in P(L)}\varepsilon_0(p) = \varepsilon_0(L)$ for every $L \in L(\Gamma)$ and that ε_0 is *sharp* if $\varepsilon_0(p_1) \cap \varepsilon_0(p_2) = 1$ for any two distinct points p_1, p_2, no matter if they are collinear or not. The next theorem generalizes a remark of Ivanov and Shpectorov [18, 2.7.2(iv)]:

Theorem 5.12. *Suppose that ε_0 is sharp and full. Then ε_0 extends to Γ and $\hat{\varepsilon}$ is the hull of ε.*

Proof. The extensibility of ε_0 can be proved as in Theorem 5.11. So, ε is an embedding. Theorem 5.2 implies that the embedding ε_X, induced by ε on $\mathrm{Res}(X)$, is dominant for every $X \in \Gamma$ of type $t(X) > 1$. Hence, by Theorem 4.4, $\hat{\varepsilon}$ is the hull of ε. qed

6 Tensor embeddings of $PG(n-1, K)$

In this section K is a given commutative field and $\Gamma := PG(n-1, K)$ for an integer $n > 2$. We shall describe two sharp but non-full linear embeddings of Γ, which are related to the subgeometry far from a point of a dual polar space of symplectic or hermitian type. As we will see later (Theorem 9.3), they are also involved in the natural projective embeddings of the dual polar spaces of those two types.

6.1 Plain tensor embeddings

Given two copies V_1 and V_2 of $V_0 := V(n, K)$ and isomorphisms α and β from V_0 to V_1 and V_2, we put

$$\varepsilon_{\alpha\otimes\beta}(X) := \langle v^\alpha \otimes v^\beta \rangle_{v\in X} \ (< V_1 \otimes V_0)$$

for every linear subspace X of V_0, and

$$V := \varepsilon_{\alpha\otimes\beta}(V_0) = \langle v^\alpha \otimes v^\beta \rangle_{v\in V_0}.$$

Then $\varepsilon_{\alpha\otimes\beta}$ is a K-linear embedding of $\Gamma = PG(V_0)$ in V. We call it the *plain tensor* embedding of Γ (also *tensor* embedding, for short). Note that, if $\{u_i\}_{i=1}^m$ is a basis of a subspace X of V_0, then

$$\{u_i^\alpha \otimes u_i^\beta\}_{i=1}^m \cup \{u_i^\alpha \otimes u_j^\beta + u_j^\alpha \otimes u_i^\beta\}_{1\leq i<j\leq m}$$

is a basis of $\varepsilon_{\alpha\otimes\beta}(X)$. Therefore, $\dim(\varepsilon_{\alpha\otimes\beta}(p)) = 1$ for every point p of Γ, $\dim(\varepsilon_\alpha(L))$ $= 3$ for every line $L \in L(\Gamma)$ and $\dim(V) = \binom{n+1}{2}$.

Clearly, the isomorphism class of $\varepsilon_{\alpha\otimes\beta}$ does not depend on the particular choice of the isomorphisms α and β. Accordingly, we will omit any record of α and β in our notation, writing $\varepsilon_\otimes$ for $\varepsilon_{\alpha\otimes\beta}$.

The embedding $\varepsilon_\otimes$ is sharp, but it is neither full nor laxly projective. In fact, for every line $L \in L(\Gamma)$, the subspaces $\varepsilon_\otimes(p)$ ($p \in P(L)$), regarded as points of $PG(V)$, form a conic O in the plane $\varepsilon_\otimes(L)$. According to Example 2.4 and recalling that $T_2(O) \cong \mathcal{Q}_4(K)$ when O is a conic (Payne and Thas [29, Chapter 3]), the lower residues of the planes of $\mathrm{Exp}(\varepsilon_\otimes)$ are isomorphic to the subgeometry of the generalized quadrangle $\mathcal{Q}_4(K)$, far from a point of $\mathcal{Q}_4(K)$. Denoted by Δ the dual of the symplectic variety $\mathcal{W}_{2n-1}(K)$ of rank n defined over K and given a point S_0 of Δ (namely, a maximal singular subspace of $\mathcal{W}_{2n-1}(K)$), the quads of $\mathrm{Far}_\Delta(S_0)$ have just the same structure as the planes of $\mathrm{Exp}(\varepsilon_\otimes)$. In fact:

Lemma 6.1. $\mathrm{Exp}(\varepsilon_\otimes) \cong \mathrm{Far}_\Delta(S_0)$.

Proof. The natural isomorphism from $V_1 \otimes V_2$ to the vectors space of square K-matrices of order n maps V onto the subspace $\mathrm{Sym}(n, K)$ of symmetric matrices. Let U be the unipotent radical of the stabilizer of S_0 in $\mathrm{Aut}(\Pi)$. The group U is isomorphic to the additive group of $\mathrm{Sym}(n, K)$. So, regarding V as an additive group, we get an isomorphism $\varphi : U \xrightarrow{\cong} V$. Given a point S of $\Phi := \mathrm{Far}_\Delta(S_0)$ and an element $X \in \mathrm{Res}_\Phi(S) \cong \Gamma$, the stabilizer U_X of X in U acts transitively on the set of points of X and the function sending X to $\varepsilon^{-1}(\varphi(U_X))$ is an isomorphism form $\mathrm{Res}_\Phi(S)$ to Γ. Hence $\mathrm{Exp}(\varepsilon_\otimes) \cong \mathrm{Far}_\Delta(S_0)$. ∎

Theorem 6.2. *If either $n > 3$ or $K \neq GF(2)$, then $\varepsilon_\otimes$ is abstractly dominant. If $n = 3$ and $K = GF(2)$, then the codomain $U(\varepsilon_\otimes)$ of the abstract hull of $\varepsilon_\otimes$ is an elementary abelian group of order 2^7 $(= 2|V|)$.*

Proof. Suppose Δ is not the dual of $\mathcal{W}_3(2)$. Then $\mathrm{Far}_\Delta(S_0)$ is simply connected (see [27]). Hence $\varepsilon_\otimes$ is dominant, by Lemma 6.1 and Corollary 3.4. When $n = 3$ and $K = GF(2)$, the conclusion follows by combining Lemma 6.1 with a result of Baumeister, Meixner and Pasini [3, Theorem 16]. qed

Denoted by $\varepsilon_{\otimes,0}$ the restriction of $\varepsilon_\otimes$ to the point-line system of Γ, by Theorems 4.3, 4.4, 6.2 and Baumeister, Meixner and Pasini [3, Theorem 16] we get the following:

Theorem 6.3. *If $K \neq GF(2)$, then $\varepsilon_{\otimes,0}$ is abstractly dominant. If $K = GF(2)$, then $U(\varepsilon_{\otimes,0})$ is elementary abelian of order 2^{2^n-1}.*

Note 6.1. The set $\varepsilon_\otimes(P(\Gamma))$ is the Veronesaen quadric of V. In view of this, we might also call $\varepsilon_\otimes$ the *Veronesean embedding.*

6.2 Twisted tensor embeddings

With V_0, V_1, V_2, α and β as in the previous subsection, let σ be an involutory automorphism of K and $K_\sigma := \{t \in K \mid t^\sigma = t\}$. Given a basis $\{u_i\}_{i=1}^n$ of V_0, let γ be the semilinear transformation sending $\sum_{i=1}^n t_i u_i$ to $\sum_{i=1}^n t_i^\sigma u_i^\beta$. (Note that $u_i^\gamma = u_i^\beta$ for $i = 1, ..., n$.) Let $f^\sigma_{\alpha,\beta} : V_0 \to V_1 \otimes V_2$ be the mapping sending $v \in V_0$ to $v^\alpha \otimes v^\gamma$. So, for every family $\{v_i\}_{i=1}^m$ of vectors of V_0 we have

$$\begin{aligned} f^\sigma_{\alpha,\beta}(\sum_i t_i v_i) &= \sum_i t_i^{1+\sigma}(v_i^\alpha \otimes v_i^\gamma) + \sum_{i\neq j} t_i t_j^\sigma (v_i^\alpha \otimes v_j^\gamma) \\ &= \sum_i a_i v_{i,i} + \sum_{i<j} b_{i,j}(v_{i,j} + v_{j,i}) + \sum_{i<j} c_{i,j}(\omega v_{i,j} + \omega^\sigma v_{j,i}) \end{aligned}$$

where $v_{i,j} := v_i^\alpha \otimes v_j^\gamma$, ω is a given element of $K \setminus K_\sigma$, $a_i := t_i^{1+\sigma}$ $(\in K_\sigma)$ and $b_{i,j}, c_{i,j} \in K_\sigma$ are such that $t_i t_j^\sigma = b_{i,j} + \omega c_{i,j}$. We also regard $V_1 \otimes V_2$ as a $2n^2$-dimensional K_σ-vector space W. Accordingly, $f^\sigma_{\alpha,\beta}$ is a (non-linear) mapping from V_0 to W. Using the symbol $\langle ... \rangle$ only to denote spans in W, $\langle\ \mathrm{Im}(f^\sigma_{\alpha,\beta})\rangle$ is the subspace V of W spanned by the independent set

$$\{u_i^\alpha \otimes u_i^\gamma\}_{i=1}^n \cup \{u_i^\alpha \otimes u_j^\gamma + u_j^\alpha \otimes u_i^\gamma\}_{i<j} \cup \{\omega(u_i^\alpha \otimes u_j^\gamma) + \omega^\sigma(u_j^\alpha \otimes u_i^\gamma\}_{i<j}.$$

Thus, $\dim(V) = \binom{n+1}{2} + \binom{n}{2} = n^2$. We define $\varepsilon^\sigma_{\alpha\otimes\beta}$ by the following clause:

$$\varepsilon^\sigma_{\alpha\otimes\beta}(X) := \langle f_{\alpha,\gamma}(v)\rangle_{v\in X}$$

for X a non-trivial proper linear subspace of V_0 and where, according to the above conventions, spans are done in V. Then $\varepsilon^\sigma_{\alpha\otimes\beta}$ is a K_σ-linear embedding of Γ in V. We call it a *twisted tensor embedding.* As in the case of plain tensor embeddings, we may freely change α and β without changing the isomorphism type of $\varepsilon^\sigma_{\alpha\otimes\beta}$. Accordingly, we will write $\varepsilon^\sigma_\otimes$ instead of $\varepsilon^\sigma_{\alpha\otimes\beta}$.

We have $\dim(\varepsilon^\sigma_\otimes(X)) = m^2$ for every $(m-1)$-element X of Γ. In particular, $\dim(\varepsilon^\sigma_\otimes(p)) = 1$ and $\dim(\varepsilon^\sigma_\otimes(L)) = 4$ for every point $p \in P(\Gamma)$ and every line $L \in L(\Gamma)$. Therefore, $\varepsilon^\sigma_\otimes$ is sharp, but it is neither full nor laxily projective. In fact, for every line $L \in L(\Gamma)$, the set $O := \{\varepsilon^\sigma_\otimes(p)\}_{p\in P(L)}$, regarded as a set of points of $PG(V)$, is a classical ovoid of the 3-space $\varepsilon^\sigma_\otimes(L)$ of $PG(V)$. According to Example 2.4 and recalling that $T_3(O) \cong \mathcal{Q}^-_5(K_\sigma)$ when the ovoid O is classical (Payne and Thas [29, Chapter 3]), the lower residues of the planes of $\mathrm{Exp}(\varepsilon^\sigma_\otimes)$ are isomorphic to the subgeometry of the generalized quadrangle $\mathcal{Q}^-_5(K_\sigma)$ far from a point of $\mathcal{Q}^-_5(K_\sigma)$. Like in the plain case, denoted by Δ the dual of the hermitian variety $\mathcal{H}_{2n-1}(K)$ of rank n defined over K and given a point S_0 of Δ, we have:

Lemma 6.4. $\mathrm{Exp}(\varepsilon^\sigma_\otimes) \cong \mathrm{Far}_\Delta(S_0)$.

Proof. Let H^+ (respectively, H^-) be the additive group of σ-hermitian (σ-anti-hermitian) square matrices of order n with entries in K. Clearly, $V \cong H^+$ and the unipotent radical U of the stabilizer of S_0 in $\mathrm{Aut}(\Delta)$ is isomorphic to H^-. However, there exists a scalar $\lambda \in K^*$ such that, for every square matrix M, we have $M \in H^+$ if and only if $M\lambda \in H^-$. Hence $V \cong U$ and the conclusion follows as in the proof of Lemma 6.1. ∎

Theorem 6.5. *If either $n > 3$ or $K \neq GF(4)$, then $\varepsilon^\sigma_\otimes$ is abstractly dominant. If $n = 3$ and $K = GF(4)$, then $U(\varepsilon^\sigma_\otimes)$ is an elementary abelian group of order 2^{11} (which is 4 times the order of V).*

Proof. When Δ is not the dual of $\mathcal{H}_5(4)$, $\mathrm{Far}_\Delta(S_0)$ is simply connected [27], whence $\varepsilon^\sigma_\otimes$ is dominant by Lemma 6.4 and Corollary 3.4. When $n = 3$ and $K = GF(4)$ the conclusion follows from [27, Theorem 1.6]. ∎

Denoted by $\varepsilon^\sigma_{\otimes,0}$ the restriction of $\varepsilon^\sigma_\otimes$ to the point-line system of Γ, theorems 4.3, 4.4 and 6.5 imply the following:

Theorem 6.6. *If $K \neq GF(4)$, then $\varepsilon^\sigma_{\otimes,0}$ is abstractly dominant.* ∎

Problem 1. Compute the hull of $\varepsilon^\sigma_\otimes$ when $n > 3$ and $K = GF(4)$.

7 Hulls of embeddings of polar spaces

In this section Γ is a classical polar space of rank $n \geq 2$ and $\varepsilon : \Gamma \to PG(V)$ is a full projective embedding of Γ. So, for a given antiautomorphism σ of the underlying division ring K of V and a suitable $e \in K$ with $ee^\sigma = 1$, $\varepsilon(\Gamma)$ is the family of linear subspaces of V that are totally isotropic for a non-degenerate trace-valued reflexive (σ, e)-sequilinear form or totally singular for a non-singular (σ, e)-pseudoquadratic form. Following Tits [35], we set

$$(1) \quad K_{\sigma,e} := \{t - t^\sigma e\}_{t\in K}.$$

We also put

$$(2) \quad \widehat{K}_{\sigma,e} := \{t \in K \mid t + t^\sigma e = 0\}.$$

A few well known properties of $K_{\sigma,e}$ and $\widehat{K}_{\sigma,e}$ are gathered in the next lemma:

Lemma 7.1. *Both $K_{\sigma,e}$ and $\widehat{K}_{\sigma,e}$ are subgroups of the additive group of K and $K_{\sigma,e} \leq \widehat{K}_{\sigma,e}$. If $\sigma = id_K$ and $e = 1$, then $K_{\sigma,e} = 0$. If $\sigma = id_K$ but $e = -1$, then $\widehat{K}_{\sigma,e} = K$. If the center $Z(K)$ of K contains an element a such that $a + a^\sigma \neq 0$, then $K_{\sigma,e} = \widehat{K}_{\sigma,e}$. In particular, if $ch(K) \neq 2$ then $K_{\sigma,e} = \widehat{K}_{\sigma,e}$.* ∎

The universal cover $\widetilde{\Delta}$ of the expansion $\Delta := \mathrm{Exp}(\varepsilon)$ is the geometry $\mathrm{Far}_\Pi(p_0)$, for a point p_0 of a polar space Π of rank $n+1$ (Cuypers and Pasini [11]). Explicitly, $\Delta = \widetilde{\Delta}/E_0$ where E_0 is the subgroup of $\mathrm{Aut}(\widetilde{\Delta})$ induced by the stabilizer in $\mathrm{Aut}(\Pi)$ of all points of Π collinear with p_0. For every point p of Δ, the canonical projection $\pi : \widetilde{\Delta} \to \widetilde{\Delta}/E_0 = \Delta$, being a covering, induces an isomorphism from $\mathrm{Res}_{\widetilde{\Delta}}(p)$ to $\mathrm{Res}_\Delta(\pi(p))$. On the other hand, $\mathrm{Res}_\Delta(\pi(p)) \cong \Gamma$, as $\Delta = \mathrm{Exp}(\varepsilon)$, whereas $\mathrm{Res}_{\widetilde{\Delta}}(p) = \mathrm{Res}_\Pi(p) \cong \mathrm{Res}_\Pi(p_0)$. Hence $\mathrm{Res}_\Pi(p_0) \cong \Gamma$. Given an isomorphism α from $\mathrm{Res}_\Pi(p_0)$ to Γ, let $W := V \oplus V(2, K)$. There exists a projective embedding $\varepsilon_\alpha : \Pi \to PG(W)$ such that, denoted by W_0 the span of $\cup_{X \in \mathrm{Res}_\Pi(p_0)} \varepsilon_\alpha(X)$ in W and by $\varepsilon_{\alpha,0}$ the embedding of $\mathrm{Res}_\Pi(p_0)$ in W_0 induced by ε_α (see Subsection 2.7), we have

$$(3) \quad \varepsilon_{\alpha,0}/P_0 \cong_{\mathrm{lin}} \varepsilon\alpha, \quad \text{where } P_0 := \mathrm{Ker}(\varepsilon_{\alpha,0}) = \varepsilon_\alpha(p_0).$$

Furthermore, we can always choose ε_α in such a way that the isomorphism (3) is realized by a linear trasformation from W_0/P_0 to V. Assuming to have chosen ε_α in that way, we have the following:

Lemma 7.2. *The image $\varepsilon_\alpha(\Pi)$ of Π consists of the linear subspaces of W that are totally isotropic or totally singular for a given (σ, e)-sequilinear or (σ, e)-pseudoquadratic form, for the same σ and e as above. Denoted by $\perp$ the orthogonality relation defined on W by that form or, in the pseudoquadratic case, by its sesquilinearization, we have $W_0 = P_0^\perp$.* ∎

Thus, W_0 is a hyperplane of W and $\widetilde{\Delta}$ is a subgeometry of the affine geometry $AG(W_0)$, the latter being regarded as the complement of W_0 in $PG(W)$. By (3), $\mathrm{Exp}(\varepsilon_{\alpha,0}) \cong \mathrm{Exp}(\varepsilon\alpha)$. However, $\mathrm{Exp}(\varepsilon\alpha) = \mathrm{Exp}(\varepsilon) = \Delta$. So, the isomorphism (3) induces an isomorphism $\mathrm{Exp}(\alpha)$ from Δ to $\mathrm{Exp}(\varepsilon_{\alpha,0})$. The composition $\pi_\alpha := \mathrm{Exp}(\alpha)\pi$ of the canonical projection $\pi : \widetilde{\Delta} \to \widetilde{\Delta}/E_0 = \Delta$ with $\mathrm{Exp}(\alpha)$ is a covering from $\widetilde{\Delta}$ to $\mathrm{Exp}(\varepsilon_{\alpha,0})$. The elements of $\mathrm{Exp}(\varepsilon_{\alpha,0})$ can be regarded as subspaces of $PG(W)$ containing P_0 but not contained in W_0. The covering π_α sends every element $X \in \widetilde{\Delta}$ to the subspace $\langle \varepsilon_\alpha(X), P_0 \rangle$ of $PG(W)$. The group E_0, regarded as a subgroup of $\mathrm{Aut}(PG(W))$, is the stabilizer of $\varepsilon_\alpha(\Pi)$ in the group of the elations of $PG(W)$ with center P_0 and axis W_0. So,

Lemma 7.3. *The group E_0 stabilizes every line L of $PG(W)$ through P_0 and, when $L \not\subseteq W_0$, it acts regularly on the set of points of $L \cap \varepsilon_\alpha(\Pi)$ different from P_0.* ∎

By Theorem 3.3 we also obtain the following:

Lemma 7.4. *$\widetilde{\Delta}$ is the expansion $\mathrm{Exp}(\tilde{\varepsilon})$ of the abstract hull $\tilde{\varepsilon}$ of ε.* ∎

Theorem 7.5. *The codomain $U(\varepsilon)$ of $\tilde{\varepsilon}$ is isomorphic to the unipotent radical of the stabilizer of p_0 in $\mathrm{Aut}(\Pi)$.*

Proof. By Lemma 7.4 and Proposition 2.5, $U(\varepsilon)$ can be recovered as a subgroup $\widetilde{G}$ of $\mathrm{Aut}(\widetilde{\Delta})$, regular on $P(\widetilde{\Delta})$. Furthermore, Δ is a subgeometry of $AG(V)$ and the natural projection of $\widetilde{\Delta}$ onto Δ maps $\widetilde{G}$ onto the additive group of V, acting on $AG(V)$ as the translation group. Turning to W_0 and P_0, the quotient W_0/P_0 is a hyperplane of the vector space W/P_0. Let E be the group of the elations of $PG(W/P_0)$ that have W_0/P_0 as the axis. As every element of E induces an automorphism on $\mathrm{Exp}(\varepsilon_{\alpha,0})$ and $\widetilde{\Delta}$ is simply connected, every element of E lifts via π_α to an automorphism of Π stabilizing P_0. This shows that $\widetilde{G}/E_0 \cong E$. The conclusion follows. ∎

Corollary 7.6. *The group $U(\varepsilon)$ is an extension of the additive group of V by E_0. In its turn, E_0 is isomorphic to either $\widehat{K}_{\sigma,e}$ or $K_{\sigma,e}$, according to whether the form that defines $\varepsilon(\Gamma)$ in V is sesquilinear or pseudoquadratic.*

Proof. The group E is isomorphic to the additive group of W_0/P_0 and $W_0/P_0 \cong V$. As $\widetilde{G}/E_0 \cong E$ (notation as in the proof of Theorem 7.5), the first claim of the Corollary follows. The second claim follows from Lemmas 7.2 and 7.3. ∎

The next theorem generalizes a result stated for $K = GF(2)$ by Ivanov and Shpectorov [18, 3.6.2].

Theorem 7.7. *The embedding ε is abstractly dominant if and only if $\varepsilon(\Gamma)$ is a quadric.*

Proof. When ε embeds Γ in $PG(V)$ as a quadric, then $\varepsilon_\alpha(\Pi)$ is also a quadric of $PG(W)$ (see Lemma 7.2) and every line of $PG(W)$ through P_0 not contained in W_0 meets $\varepsilon_\alpha(\Pi)$ in exactly two points, one of which is equal to P_0. Hence $E_0 = 1$ and $\Delta \cong \widetilde{\Delta}$. So, Δ is simply connected and, by Corollary 3.4, ε is dominant. Conversely, suppose that ε is dominant. Then $E_0 = 1$ by Corollary 7.6. Therefore, by the second claim of Corollary 7.6, either $\widehat{K}_{\sigma,e} = 0$ (when $\varepsilon(\Gamma)$ is defined by a sesquilinear form) or $K_{\sigma,e} = 0$ (if $\varepsilon(\Gamma)$ arises from a pseudoquadratic form). In any case, $\sigma = \mathrm{id}_K$, $e = 1$ and $\varepsilon(\Gamma)$ is a quadric. ∎

Lemma 7.8. *Suppose that either* $\mathrm{ch}(K) \neq 2$ *or* $\sigma_{|Z(K)} \neq id_{Z(K)}$. *Then E_0 is the commutator subgroup of $U(\varepsilon)$.*

Proof. By Theorem 7.5 and Corollary 7.6, $U(\varepsilon) \cong U$, where U stands for the unipotent radical of the stabilizer of p_0 in $\mathrm{Aut}(\Pi)$, $E_0 \trianglelefteq U$ and U/E_0 is the additive group of V. In view of the hypotheses made on K and σ, we may assume that Π arises from a sesquilinear form. Regarded Π in that way, the elements of U are the pairs $(x, (x_i)_{i=1}^{2n})$ where $(x_i)_{i=1}^{2n} \in V$ and $x \in K$ is such that

$$(*) \quad x + e^{-1}x^\sigma + \sum_{i=1}^{n} x_{n+i}x_i^\sigma + \sum_{i=1}^{n} x_i e^{-1} x_{n+i}^\sigma = 0.$$

Products are as follows: $(x, (x_i)_{i=1}^{2n}) \cdot (y, (y_i)_{i=1}^{2n}) = (z, (x_i + y_i)_{i=1}^{2n})$ where

$$z = x + y - \sum_{i=1}^{n} x_{n+i}y_i^\sigma - \sum_{i=1}^{n} x_i e^{-1} y_{n+i}^\sigma.$$

The subgroup $E_0 \trianglelefteq U$ consist of the pairs $(x, (0, 0, ..., 0))$ where, according to $(*)$, $x + e^{-1}x^\sigma = 0$. The function f_σ sending $(x, (0, 0, ..., 0))$ to x^σ is an isomorphism from E_0 to $\widehat{K}_{\sigma,e}$. The commutator subgroup U' of U is contained in E_0 and is equal to $f_\sigma^{-1}(K_{\sigma,e})$. However, the hypotheses made on K and σ force $K_{\sigma,e} = \widehat{K}_{\sigma,e}$ (see Lemma 7.1). Hence $U' = E_0$. ■

Corollary 7.6 and Lemma 7.8 imply the following:

Theorem 7.9. *Suppose that either* $\mathrm{ch}(K) \neq 2$ *or* $\sigma_{|Z(K)} \neq id_{Z(K)}$. *Then* ε *is its own abelian hull.* ■

As a corollary, we obtain the following well known result of Tits [35, 8.6]:

Corollary 7.10. *Under the hypotheses of Theorem* 7.9*,* ε *is linearly dominant.* ■

Example 7.1. Suppose that $\varepsilon(\Gamma)$ arises from an alternating form. Then $E_0 \cong \widehat{K}_{\sigma,e} = K$ and $U(\varepsilon)$ is an extension of V by the additive group of K (Corollary 7.6). By Lemma 7.8, if $\mathrm{ch}(K) \neq 2$ then $U(\varepsilon)' = K$. On the other hand, when $\mathrm{ch}(K) = 2$ then $U(\varepsilon)$ is commutative. In this case Γ can also be embedded as a quadric in $PG(\tilde{V})$, with $\widetilde{V} := V(1, K) \oplus V$. The latter embedding dominates ε and is dominant by Theorem 7.7. Hence it is the hull of ε. Namely, $U(\varepsilon) \cong \widetilde{V}$.

8 Embeddings of grassmannians

8.1 Grassmann embeddings of $PG(n, K)$

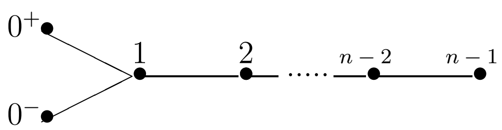
We recall that, given $\Pi := PG(n, K)$ with $n > 2$ and a positive integer $m < n - 1$, the *m-grassmannian* $\mathrm{gr}_m(\Pi)$ of Π is the point-line geometry with the m-elements of Π as points, the flags of type $\{m - 1, m + 1\}$ as lines and the incidence relation inherited from Π. In particular, $\mathrm{gr}_1(\Pi)$ is the *line-grassmannian* of Π.

Note 8.1. We warn that many authors, referring to vector-dimensions instead of projective dimensions, replace m by $m + 1$ in the above definitions.

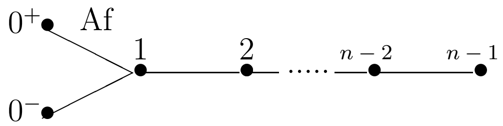
It is well known (Wells [38]) that, when K is a field, $\mathrm{gr}_m(\Pi)$ admits a full projective embedding in $PG(\wedge^{m+1}V)$ where $V := V(n + 1, K)$. We shall call that embedding ε_{gr_m}. Wells [38] also proves that ε_{gr_m} is linearly dominant. In the case of $m = 1$, a stronger result holds:

Theorem 8.1. *The embedding* ε_{gr_1} *is abstractly dominant.*

Proof. We firstly give another description of $\mathrm{Exp}(\varepsilon_{gr_1})$. Let Δ be the building of type $D_{n+1}(K)$, with types $0^+, 0^-, 1, 2, ..., n - 1$ as follows:



Given an element A of Δ of type 0^+ (when n is odd) or 0^- (when n is even), the geometry $\Phi := \mathrm{Far}_\Delta(A)$ belongs to the following diagram of rank $n+1$, where the label Af stands for the class of affine planes:



The elements of type 0^+ are taken as points and those of type 1 as lines. The $\{0^-, 2\}$-flags are regarded as planes. We denote by $\mathrm{Sh}(\Phi)$ the 0^+-shadow geometry of Φ (namely, the 0^+-grassmann geometry of Φ, according to the terminology of [26]). Let Σ be the $\{0, 1, 2\}$-truncation of $\mathrm{Sh}(\Phi)$, formed by the elements of Φ of type 0^+ and 1 and by the flags of type $\{0^-, 2\}$. We have $\mathrm{Res}_\Sigma(P) \cong \mathrm{gr}_1(\Pi)$ for every point P of Σ. In fact:

Lemma 8.2. $\Sigma \cong \mathrm{Exp}(\varepsilon_{gr_1})$.

Proof. Let U be the unipotent radical of the stabilizer of A in $\mathrm{Aut}(\Delta)$. Then U is isomorphic to the additive group of $\wedge^2 V$ and acts regularly on the set of points of Σ. For every point P of Σ, we have $\mathrm{Res}_\Sigma(P) \cong \mathrm{gr}_1(\mathrm{Res}_\Delta(P))$ ($\cong \mathrm{gr}_1(\Pi)$) and the stabilizer U_X in U of a line or a plane $X > P$ of Σ is transitive on the set of points of X different from P. Furthermore, the isomorphism $\varphi : U \xrightarrow{\cong} \wedge^2 V$ can be chosen in such a way that, for every $X \in \mathrm{Res}_\Sigma(P)$, $\varphi(U_X)$ belongs to the image of ε_{gr_1} and the function sending X to $\varepsilon_{gr_1}^{-1}(\varphi(U_X))$ is an isomorphism from $\mathrm{Res}_\Sigma(P)$ to $\mathrm{gr}_1(\Pi)$. The conclusion is now evident. ∎

Lemma 8.3. *The geometry Σ is simply connected.*

Proof. Φ is simply connected (see [27]; also Munemasa and Shpectorov [23] and Munemasa, Pasechnik and Shpectorov [24] for the finite case). Consequently $\mathrm{Sh}(\Phi)$ is simply connected by [26, Theorem 12.64]. All lower residues of elements of $\mathrm{Sh}(\Phi)$ of type at least 3 are simply connected as well. Hence, by repeatedly applying Theorem 12.64 of [26] to closed paths of the collinearity graph of $\mathrm{Sh}(\Phi)$ and of lower residues of elements of $\mathrm{Sh}(\Phi)$, we see that every closed path of the collinearity graph of Σ splits in subpaths each of which is contained in a plane. Therefore, again by Theorem 12.64 of [26], Σ is simply connected. ∎

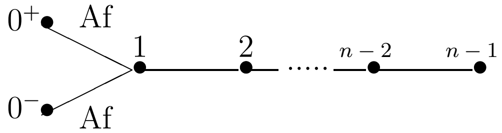
End of the proof of Theorem 8.1. By Lemmas 8.2 and 8.3, $\mathrm{Exp}(\varepsilon_{gr_1})$ is simply connected. By Corollary 3.4, ε_{gr_1} is dominant. ∎

Problem 2. What about ε_{gr_2}? Consider the case of $n = 5$ and $K = GF(2)$, to begin with. By [10, page 191], $\mathrm{Exp}(\varepsilon_{gr_2})$ is a 2-fold quotient of $\mathrm{Far}_\Phi(A)$ for a 5-element A of the building Φ of type $E_6(2)$, where types are given as in Subsection 10. The geometry $\mathrm{Far}_\Phi(A)$ is likely to be simply connected. If so, then we would get $U(\varepsilon_{gr_2}) = 2^{1+20}$, by Corollary 3.4.

8.2 A grassmann embedding of $AG(n, K)$

Regarded $AG(n, K)$ as a subgeometry of $\Pi = PG(n, K)$, we can form the grassmannian $\mathrm{gr}_1(AG(n, K))$ of $AG(n, K)$. Clearly, $\mathrm{gr}_1(AG(n, K))$ is a subgeometry of $\mathrm{gr}_1(\Pi)$ and ε_{gr_1} induces on it a full projective embedding $\varepsilon_{gr_1}^{Af}$ in $PG(\wedge^2 V)$.

Given a building Δ of type $D_{n+1}(K)$ as in the proof of Theorem 8.1, we now take a flag $\{A, B\}$ of type $\{0^+, 0^-\}$ and put $\Phi := \mathrm{Far}_\Delta(\{A, B\})$. The diagram of Φ is as follows:



Let Σ be the $\{0, 1, 2\}$-truncation of $\mathrm{Sh}(\Phi)$, with the elements of type 0^+ and 1 regarded as points and lines respectively. Then,

Lemma 8.4. $\Sigma \cong \mathrm{Exp}(\varepsilon_{gr_1}^{Af})$.

Proof. The proof is the same as in Lemma 8.2. We leave the details for the reader. We only remark that the commutator subgroup of the unipotent radical of the stabilizer of $\{A, B\}$ in $\mathrm{Aut}(\Delta)$ corresponds to $\wedge^2 V$ and acts regularly on the set of points of Σ. ■

Theorem 8.5. *If $K \neq GF(2)$, then $\varepsilon_{gr_1}^{Af}$ is abstractly dominant. When $K = GF(2)$, then $U(\varepsilon_{gr_1}^{Af})$ is elementary abelian of order 2^{2^n-1}.*

Proof. Suppose $K \neq GF(2)$. Then Φ is simply connected [27] (also Baumeister and Stroth [2, Theorem 6.5(2)], in the finite case). Hence Σ is simply connected (same argument as in the proof of Lemma 8.3) and we obtain the conclusion by Corollary 3.4.

On the other hand, let $K = GF(2)$. Then the universal 2-cover $\widetilde{\Phi}$ of Φ can be built inside a Coxeter complex of type D_{2^n} and the deck group of the covering is elementary abelian of order $2^{2^n-\binom{n+1}{2}-1}$ (Baumeister, Meixner and Pasini [3]; also Baumeister and Stroth [2, Theorem 6.5(1)]). We shall now prove that the shadow geometry $\mathrm{Sh}(\widetilde{\Phi})$ of the universal 2-cover $\widetilde{\Phi}$ of Φ is 2-simply connected.

By Theorem 12.64 of [26], $\mathrm{Sh}(\widetilde{\Phi})$ is simply connected, as such is $\widetilde{\Phi}$. When $n = 3$, then all residues of $\widetilde{\Phi}$ of rank 3 are simply connected. Hence, when $n = 3$, $\mathrm{Sh}(\widetilde{\Phi})$ is 2-simply connected. We go on by induction. Let $n > 3$. Then the residues of the elements of $\mathrm{Sh}(\widetilde{\Phi})$ corresponding to elements of $\widetilde{\Phi}$ of type $i = 3, 4, ..., n-1$ are 2-simply connected by the inductive hypothesis. The residues of the remaining elements of $\mathrm{Sh}(\widetilde{\Phi})$ are either affine geometries or direct sums of affine geometries and projective geometries, hence they are 2-simply connected. Thus, all residues of elements of $\mathrm{Sh}(\widetilde{\Phi})$ are 2-simply connected. This forces every 2-covering of $\mathrm{Sh}(\widetilde{\Phi})$ to be a covering. However, $\mathrm{Sh}(\widetilde{\Phi})$ is simply connected. Hence its is 2-simply connected. Moreover, $\mathrm{Sh}(\widetilde{\Phi})$ is a 2-cover $\mathrm{Sh}(\Phi)$. Therefore, $\mathrm{Sh}(\widetilde{\Phi})$ is the universal 2-cover of $\mathrm{Sh}(\Phi)$, whence it is 2-simply connected.

The residues of the flags of $\mathrm{Sh}(\Phi)$ of corank at least three and containing some elements of type $0, 1$ or 2 are simply connected. So, we can apply Theorem 1 of [25], obtaining that the universal cover $\widetilde{\Sigma}$ of Σ is the $\{0, 1, 2\}$-truncation of $\mathrm{Sh}(\widetilde{\Phi})$.

Consequently, $\widetilde{\Sigma}$ can be constructed inside a Coxeter complex of type D_{2^n}. Moreover, denoted by W the Coxeter group of that type, the subgroup $U \cong \wedge^2 V$ of $\mathrm{Aut}(\Sigma)$ lifts to $O_2(W)$ ($\cong 2^{2^n-1}$). The conclusion now follows from the isomorphism $\Sigma \cong \mathrm{Exp}(\varepsilon_{gr_1}^{Af})$ and Theorem 3.3. ∎

9 Projective embeddings of dual polar spaces

9.1 The embeddings considered in this section

In the sequel, F is a given finite field and Π is one of the following polar spaces of rank $n \geq 3$:

1) the symplectic variety $\mathcal{W}_{2n-1}(F)$ arising from a non-degenerate alternating form over $V(2n, F)$;
2) the hermitian variety $\mathcal{H}_{2n-1}(F)$ arising from a non-degenerate unitary form over $V(2n, F)$;
3) the quadric $\mathcal{Q}_{2n}(F)$ defined by a non-singular quadratic form of Witt index n in $V(2n+1, F)$;
4) the quadric $\mathcal{Q}^-_{2n+1}(F)$ defined by a non-singular quadratic form of Witt index n in $V(2n+2, F)$.

Let Γ be the dual of Π and $\Sigma := (P(\Gamma), L(\Gamma))$ the point-line system of Γ. For each of the above four cases there exists a natural full projective embedding $\varepsilon_0 : \Sigma \to PG(m, K)$, which we sketchily describe below. We refer to Cooperstein and Shult [9], Cooperstein [7] and [8] and Wells [38] for more details.

Case 1. $\Pi = \mathcal{W}_{2n-1}(F)$**.** Here $K = F$ and $m + 1 = \binom{2n}{n} - \binom{2n}{n-2}$. Put $V_1 := V(2n, F)$, $V_2 := \wedge^n V_1$ and $\varepsilon_{gr} := \varepsilon_{gr_{n-1}}$, where $\varepsilon_{gr_{n-1}}$ is the embedding of the $(n-1)$-grassmannian $\mathrm{gr}_{n-1}(PG(V_1))$ in $PG(V_2)$ (Section 8). The points of Σ are points of $\mathrm{gr}_{n-1}(PG(V_1))$ and the injective function sending every non-maximal singular subspace X of Π to the flag $\{X, X^\perp\}$ of $PG(V_1)$ sends the lines of Σ to lines of $\mathrm{gr}_{n-1}(PG(V_1))$. So, Σ is a subgeometry of $\mathrm{gr}_{n-1}(PG(V_1))$ and, denoted by V the subspace of V_2 spanned by $\{\varepsilon_{gr}(p)\}_{p \in P(\Gamma)}$, ε_{gr} induces a full projective embedding $\varepsilon_0 : \Sigma \to PG(V)$. The number $m + 1 = \binom{2n}{n} - \binom{2n}{n-2}$ is the dimension of V (Cooperstein [7]).

When $F \neq GF(2)$, ε_0 is the universal projective embedding of Σ (Cooperstein [7]; also Cooperstein and Shult [9]). Hence ε_0 is linearly dominant, but probably it is not abstractly dominant (compare Subsection 9.4, Problem 3).

Suppose $F = GF(2)$. Then ε_0 is not even linearly dominant. Its linear hull embeds Σ in $PG(d, 2)$ where $d + 1 = (2^n + 1)(2^{n-1} + 1)/3$ (Li [21]). Ivanov [15] has proved that the codomain $U(\varepsilon_0)$ of the abstract hull of ε_0 is non-abelian. In the smallest case ($n = 3$), the commutator subgroup of $U(\varepsilon_0)$ has order 2 (Yoshiara [39], Ivanov [15]; see also Theorem 9.8 of this paper).

Case 2. $\Pi = \mathcal{H}_{2n-1}(F)$. Now F is a quadratic extension of K and $m+1 = \binom{2n}{n}$. The embedding $\varepsilon_0 : \Sigma \to PG(m,K)$ can be described as follows. With $V_1 := V(2n,F)$, $V_2 := \wedge^n V_1$ and $\varepsilon_{gr} : \mathrm{gr}_{n-1}(PG(V_1)) \to PG(V_2)$ as in the previous paragraph, we now have $\langle \varepsilon_{gr}(p)\rangle_{p\in P(\Gamma)} = V_2$ (Cooperstein [8]). Hence ε_{gr} induces a lax projective embedding of Σ in $PG(V_2)$. However, V_2 admits a basis $B = \{b_i\}_{i=0}^m$ such that, denoted by V the vector space formed by the linear combinations $\sum_i t_i b_i$ with all scalars t_i in K, $\varepsilon_{gr}(p) \cap V$ is a 1-dimensional linear subspace of V for every point $p \in P(\Gamma)$ (Cooperstein [8]). Thus, we get a full embedding $\varepsilon_0 : \Sigma \to PG(V)$.

When $F \neq GF(4)$, ε_0 is the universal projective embedding of Σ (Cooperstein [8]; also Cooperstein and Shult [9]). When $F = GF(4)$, the linear hull of ε_0 embeds Σ in $PG(d,2)$ where $d = (4^n+2)/3$ (Li [22]). Furthermore, when $n = 3$, the commutator subgroup of the codomain $U(\varepsilon_0)$ of the abstract hull of ε_0 has order 2 (Ivanov [15]; see also Theorem 9.8).

Case 3. $\Pi = \mathcal{Q}_{2n}(F)$. Here $K = F$, $m+1 = 2^n$ and ε_0 is the well known *spin-embedding* of Π (Wells [38]). When $\mathrm{ch}(F) \neq 2$, ε_0 is the universal projective embedding of Σ (Wells [38]; also Cooperstein and Shult [9]). However, ε_0 it is not abstractly dominant (see Theorem 9.12), but it extends to Γ and its extension is abstractly dominant (Theorem 9.11). When $\mathrm{ch}(F) = 2$, then $\mathcal{Q}_{2n}(F) \cong \mathcal{S}_{2n-1}(F)$ and we are led back to Case 1.

Case 4. $\Pi = \mathcal{Q}^-_{2n+1}(F)$. In this case K is a quadratic extension of F and $m+1 = 2^n$. The polar space Π is a subgeometry of $\Pi' := \mathcal{Q}^+_{2n+1}(K)$, naturally embedded in Π' by the lax Baer embedding of $PG(2n+1,F)$ in $PG(2n+1,K)$. Thus, Γ can also be regarded as a subgeometry of the dual Γ' of Π'. The half-spin geometry of Γ' (see Section 10) admits a full projective embedding in $PG(2^n-1,K)$ (Wells [38]) and the latter induces a full projective embedding $\varepsilon_0 : \Sigma \to PG(2^n-1,K)$, which is in fact the universal projective embedding of Σ (Cooperstein and Shult [9]).

In the next subsection we prove that, in each of the above four cases, ε_0 extends to Γ. In Subsection 9.3 we describe the embedding induced on point-residues of Γ by the extension of ε_0. Finally (Subsections 9.4 and 9.5), we give more information on the cases of $\Pi = \mathcal{W}_5(F), \mathcal{H}_5(F)$ and $\mathcal{Q}_6(F)$.

9.2 Extensibility of ε_0

Theorem 9.1. *In each of the previous four cases, ε_0 extends to Γ.*

Proof. Let G be the setwise stabilizer of $\varepsilon_0(\Sigma)$ in $P\Gamma L(m+1,K)$. Every element of G induces an automorphism of Σ and the kernel of the homomorphism thus defined from G to $\mathrm{Aut}(\Sigma)$ is the automorphism group $\mathrm{Aut}(\varepsilon_0)$ of ε_0, namely the point-wise stabilizer of $\varepsilon_0(\Sigma)$ in $P\Gamma L(m+1,K)$. Thus, we may regard the quotient $\overline{G} := G/\mathrm{Aut}(\varepsilon_0)$ as a subgroup of $\mathrm{Aut}(\Sigma)$. On the other hand, the elements of Γ can be recovered in Σ as the convex closures of pairs of points at non-maximal distance (Brouwer and Wilbrink [5]). So, every automorphism of Σ is induced by a unique automorphism of Γ. Accordingly, $\overline{G}$ can be regarded as a subgroup of $\mathrm{Aut}(\Gamma)$. In each of the above four cases, the following is straightforward:

(1) $\overline{G}$ acts transitively on the set of triples (A, p, L) where A is an $(n-1)$-element of Γ, $p \in P(A)$ and $L \in L(p) \setminus L(A)$.

Put $\varepsilon(X) := \langle \varepsilon_0(p) \rangle_{p \in P(X)}$, as in Subsection 2.5. We shall prove that ε satisfies (E1) (whence it is an embedding). In each of the four cases under consideration, for every $(n-1)$-element A of Γ, ε_0 induces on the point-line system of $\mathrm{Res}_\Gamma(A)$ an embedding defined in the same way as ε_0, but with a codomain of dimension less than $V = \mathrm{cod}(\varepsilon_0)$. Therefore,

(2) $\varepsilon(A) < V$ for every $(n-1)$-element A of Γ.

We shall now prove the following:

(3) for every $(n-1)$-element A of Γ and every point $p \in P(\Gamma)$, if $\varepsilon_0(p) \leq \varepsilon(A)$, then $p < A$.

Suppose the contrary and let p, A be a counterexample to (3). So, $\varepsilon_0(p) \leq \varepsilon(A)$ but $p \notin P(A)$. By well known properties of dual polar spaces (Brouwer and Wilbrink [5]), A contains a unique point p' collinear with p. Let L the line on p and p'. Then $\varepsilon_0(L) = \langle \varepsilon_0(p), \varepsilon_0(p') \rangle$. Hence $\varepsilon_0(L) \leq \varepsilon(A)$, as $\varepsilon_0(p) \leq \varepsilon(A)$. However, $L \not< A$. So, A, p' and L are as in (1). By (1), $\varepsilon(A) \geq \varepsilon_0(L)$ for every line L. Hence $\varepsilon(A) = V$, contrary to (2). Claim (3) is proved.

By (3), if $\varepsilon(X) \leq \varepsilon(Y)$ then $P(X) \subseteq \cap(P(A) \mid A \geq Y,\ t(A) = n-1)$. However, $\cap(P(A) \mid A \geq Y,\ t(A) = n-1) = P(Y)$, by well known properties of dual polar spaces. Hence $P(X) \subseteq P(Y)$, which forces $X \leq Y$ by (PS) (which holds in Γ). So, ε satisfies (E1). ∎

We denote by ε the extension of ε_0 to Γ, as in the proof of Theorem 9.1, and by $\tilde{\varepsilon}_0$ the abstract hull of ε_0. By Theorem 4.1, $\tilde{\varepsilon}_0$ also extends to Γ. We denote its extension by $\hat{\varepsilon}$. We recall that $\hat{\varepsilon}$ is abstractly dominant (Theorem 4.1). Theorem 4.3 and the 2-simple connectedness of polar spaces imply the following:

Corollary 9.2. *The geometry* $\mathrm{Exp}(\hat{\varepsilon})$ *is the universal* 2*-cover of* $\mathrm{Exp}(\varepsilon)$. ∎

9.3 Embeddings induced on point-residues

In the sequel, $V = V(m+1, K)$ is the codomain of ε_0 and, according to the notation introduced at the end of the previous subsection, $\varepsilon : \Gamma \to PG(V)$ is the extension of ε_0 to Γ. Given a point $p \in P(\Gamma)$, let V_p be the subspace of V spanned by $\{\varepsilon(L)\}_{L \in L(p)}$ and $\varepsilon_p : \mathrm{Res}_\Gamma(p) \to V_p$ the embedding induced by ε on $\mathrm{Res}_\Gamma(p)$ (see Subsection 2.7). We put $W := V_p/\varepsilon(p)$ and denote by η the reduction $\varepsilon_p/\varepsilon(p)$ of ε_p. So, η is an embedding of the projective geometry $\mathrm{Res}_\Gamma(p) \cong PG(W_0)$, where $W_0 := V(n, F)$. Clearly, η is linear and, as $\dim(\varepsilon(L)) = 2$ for every line L, $\dim(\eta(L)) = 1$ for every line $L > p$.

In Cases 3 and 4 of Subsection 9.1, we also have $\dim(\varepsilon(S)) = 3$ for every quad S. (We recall that the 2-elements of a dual polar space are usually called *quads*.) Hence $\dim(\eta(S)) = 2$ for every quad $S > p$. Thus, η is a lax projective embedding.

Furthermore, η is full in Case 3. By Theorem 5.8, $\dim(W) = n$ in both cases 3 and 4. In Case 3, $F = K$, $W = W_0$ and η is just an isomorphism from $\mathrm{Res}_\Gamma(p)$ to $PG(W)$ (whence $\mathrm{Exp}(\eta) \cong AG(W)$). In Case 4, K is a quadratic extension of F and η embeds $\mathrm{Res}_\Gamma(p)$ in $PG(W)$ as a Baer subgeometry. In any case, η is dominant, namely $U(\eta) = W$ (see the second claim of Theorem 5.8). Cases 1 and 2 are more interesting.

Theorem 9.3. *The embedding $\eta : PG(W_0) \to PG(W)$ is a plain tensor embedding in Case* 1 *and a twisted tensor embedding in Case* 2.

Proof. Assume we are in Case 1. Choose a maximal singular subspace p' of Π disjoint from p. Every line X of Γ on p is a hyperplane of the projective geometry $\mathrm{Res}_\Pi(p) \cong PG(W_0^*)$, where W_0^* stands for the dual of W_0. Moreover, $X^\perp \cap p'$ is a point of $\mathrm{Res}_\Pi(p')$ and, regarded X as a singular subspace of Π, the span p_X of $X \cup (X^\perp \cap p')$ in Π is a maximal singular subspace of Π, namely a point of Γ. We have $\varepsilon(X) = \varepsilon_{gr}(\{X, X^\perp\}) = \langle \varepsilon_{gr}(p), \varepsilon_{gr}(p_X)\rangle = \langle \varepsilon(p), \varepsilon(p_X)\rangle$. Hence $\eta(X) = \langle \varepsilon(p), \varepsilon(p_X)\rangle/\varepsilon(p)$. It is now clear that η is isomorphic to the embedding sending X to $\varepsilon(p_X)$. In its turn, the latter is isomorphic to $\varepsilon_{\alpha\otimes\beta}$ where we may take the identity of W_0 as α and β is an isomorphisms from W_0 to W_0^* sending every point X of $PG(W_0)$ to the point $X^\perp \cap p'$ of $PG(W_0^*)$.

Turning to Case 2, given a maximal singular subspace p' of Π disjoint from p, the polarity of $PG(V_1)$ associated to Π induces an isomorphism from $\mathrm{Res}_\Gamma(p)$ to $\mathrm{Res}_\Pi(p') \cong PG(W_0^*)$ as in Case 1, but that isomorphism is now induced by a semilinear but non-linear mapping $\gamma : W_0 \to W_0^*$. By an argument similar to that used in Case 1, one can see that $\eta \cong \varepsilon^\sigma_{\alpha\otimes\beta}$, where α is the identity automorphism of W_0, β is the linear part of γ and σ is the involutory automorphism of F. ∎

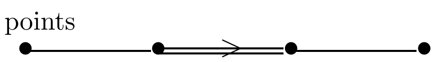
9.4 On Cases 1 and 2 with n=3

In this subsection Π is $\mathcal{W}_5(F)$ or $\mathcal{H}_5(F)$. As at the end of Subsection 9.2, ε is the extension of ε_0 to Γ, $\tilde{\varepsilon}$ and $\tilde{\varepsilon}_0$ are the abstract hulls of ε and ε_0, and $\hat{\varepsilon}$ is the extension of $\tilde{\varepsilon}_0$ to Γ.

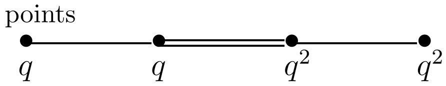
Theorem 9.4. *$\tilde{\varepsilon} = \hat{\varepsilon}$.*

Proof. In the two considered cases, for every $(n-1)$-element A of Γ, the image of $\mathrm{Res}_\Gamma(A)$ via ε_0 is a quadric. Precisely, when $\Gamma = \mathcal{W}_5(F)$, ε_0 embeds $\mathrm{Res}_\Gamma(A)$ as $\mathcal{Q}_4(K)$ in $PG(4, K)$ whereas, when $\Gamma = \mathcal{H}_5(F)$, ε_0 embeds $\mathrm{Res}_\Gamma(A)$ as $\mathcal{Q}_5^-(K)$ in $PG(5, K)$. Therefore, denoted by ε_A the embedding of $\mathrm{Res}_\Gamma(A)$ induced by ε, the expansion $\mathrm{Exp}(\varepsilon_A)$ is simply connected, by Theorem 7.7 and Corollary 3.4. The conclusion follows from Theorem 4.4. ∎

Our next goal is to describe $\tilde{\varepsilon}$ when K is a prime field. In view of that, we need a few preliminary remarks. Let Φ be the building of type $F_4(q)$ or ${}^2E_6(q)$, where $q = |K|$ (but we do not yet assume that q is prime). Points are taken as follows:

$(F_4(q))$ 

$({}^2E_6(q))$



We also put $p := \mathrm{ch}(K)$ (so, q is a power of p). Let a be a point of Φ and $U := O_p(G_a)$, where G_a is the stabilizer of a in $\mathrm{Aut}(\Phi)$. The commutator subgroup U' of U has order q and U/U' is elementary abelian of order q^{m+1}, where $m+1 = 14$ or 20 according to whether Φ is of type $F_4(q)$ or ${}^2E_6(q)$. Also, U is the kernel of the action of G_a on $\mathrm{Res}_\Phi(a)$ whereas U' is the elementwise stabilizer of the points collinear with a. Furthermore, U' defines a quotient of $\mathrm{Far}_\Phi(a)$.

Lemma 9.5. $\mathrm{Far}_\Phi(a)/U' \cong \mathrm{Exp}(\varepsilon)$.

Proof. As noticed above, U/U' is isomorphic to the additive group of the codomain $V = V(m+1, K)$ of ε. Furthermore, a K-vector space structure V^* can be put on U/U' in such a way that, for every line x of Φ on a, the elementwise stabilizer $\varepsilon_0^*(x)$ of $\mathrm{Res}_\Phi^-(x)$ in U/U' is a hyperplane of V^* and, for every plane X of Φ on a, the elementwise stabilizer $\varepsilon_0^*(X)$ of $\mathrm{Res}_\Phi^-(X)$ in U/U' is a subspace of V^* of codimension 2. Thus, we get a projective embedding ε_0^* of Σ in V, regarded as the dual of V^*. However, Σ admits just one projective embedding in $PG(V)$. When $q \neq 2$ the uniqueness of that embedding follows from the universality of ε_0. When $q = 2$, the conclusion follows from the fact that the $(m+2)$-dimensional G_a/U-module associated to the linear hull $\tilde{\varepsilon}_0$ of ε_0 admits just one $m+1$-dimensional factor. So, in any case, $\varepsilon_0^* = \varepsilon_0$.

By the above, $\mathrm{Aut}(\mathrm{Far}_\Phi(a)/U')$ and $\mathrm{Aut}(\mathrm{Exp}(\varepsilon))$ are isomorphic to the same extension of V by $G_a/U' \cong \mathrm{Aut}(\mathrm{Res}_\Phi(a))$. So, $\mathrm{Aut}(\mathrm{Far}_\Phi(a)/U') \cong \mathrm{Aut}(\mathrm{Exp}(\varepsilon))$. The isomorphism $\mathrm{Far}_\Phi(a)/U' \cong \mathrm{Exp}(\varepsilon)$ follows by comparing parabolics. ∎

Lemma 9.6. *The geometry Γ admits an embedding η in the group U. We have $\mathrm{Exp}(\eta) = \mathrm{Far}_\Phi(a)$ and η dominates ε.*

Proof. The projection of $\mathrm{Far}_\Phi(a)$ onto $\mathrm{Far}_\Phi(a)/U$ satisfies the hypotheses of condition (2) of Proposition 2.8. The conclusion follows from that proposition and Lemma 9.5. ∎

Lemma 9.7. *All the following hold for the commutator subgroup $U(\varepsilon)'$ of the codomain $U(\varepsilon)$ of $\tilde{\varepsilon}$:*

(1) *$U(\varepsilon)'$ is elementary abelian of exponent p and, regarded as a $GF(p)$-vector space, it has dimension at most h^2, where h is the positive integer such that $q = p^h$;*

(2) *$U(\varepsilon)' \leq Z(U(\varepsilon))$;*

(3) *$U(\varepsilon)' = [\tilde{\varepsilon}(x), \tilde{\varepsilon}(y)]$ for any two points $x, y \in P(\Gamma)$ at distance 3 in the collinearity graph of Γ.*

Proof. Note that, as $U(\varepsilon) = \langle \tilde{\varepsilon}(x) \rangle_{x \in P(\Gamma)}$, $U(\varepsilon)' = \langle [\tilde{\varepsilon}(x), \tilde{\varepsilon}(y)] \rangle_{x,y \in P(\Gamma)}$. Let x and y be points of Γ at distance at most 2. Given a quad Q of Γ on x and y, $\tilde{\varepsilon}$ induces on $\mathrm{Res}_\Gamma(Q)$ an embedding $\tilde{\varepsilon}_Q$ which dominates the embedding ε_Q induced by ε. However, $\varepsilon(Q)$ embeds $\mathrm{Res}_\Gamma(Q)$ as a quadric in its codomain. Hence ε_Q is dominant by Theorem 7.7. Therefore $\tilde{\varepsilon}_Q = \varepsilon_Q$. As ε_Q is abelian, $\tilde{\varepsilon}_Q$ is abelian and, consequently, $[\tilde{\varepsilon}(x), \tilde{\varepsilon}(y)] = 1$.

Suppose now x and y have distance 3 and let $x_0 = x, x_1, x_2, x_3 = y$ be a path from x to y in the collinearity graph of Γ. Let z be any point on the line through x_2 and x_3 $(= y)$. Given an element $w \in \tilde{\varepsilon}(z)$, we can always find an element $v \in \tilde{\varepsilon}(x_2)$ such that $vw \in \tilde{\varepsilon}(x_3)$. By the above, $[u, v] = 1$ for any $u \in \tilde{\varepsilon}(x_0)$. Moreover, $[v, w] = 1$. Thus, $[u, vw] = [u, v]$. Therefore, $[\tilde{\varepsilon}(x), \tilde{\varepsilon}(z)] = [\tilde{\varepsilon}(x), \tilde{\varepsilon}(y)]$. However, the set of points at distance at most 2 from x is a hyperplane of Γ. Hence we have $[\tilde{\varepsilon}(x), \tilde{\varepsilon}(z)] = [\tilde{\varepsilon}(x), \tilde{\varepsilon}(y)]$ for any two collinear points y, z at distance 3 from x. As the geometry $\mathrm{Far}_\Gamma(x)$ is connected, we also have $[\tilde{\varepsilon}(x), \tilde{\varepsilon}(y)] = [\tilde{\varepsilon}(x), \tilde{\varepsilon}(z)]$ for any two points at distance 3 from x, let them be collinear or not. We now recall that $\mathrm{Aut}(\Gamma)$ acts primitively on the set of points of Γ. Accordingly, the graph with the points of Γ as vertices and the relation 'having distance three' as the adjacency relation, is connected. Consequently, the group $[\tilde{\varepsilon}(x), \tilde{\varepsilon}(y)]$ does not depend on the choice of the pair $\{x, y\}$ of points at distance 3. Thus, (3) is proved.

Given a point z of Γ, let x be a point collinear with z and y a point at distance 2 from z and 3 from x. Then $[\tilde{\varepsilon}(z), \tilde{\varepsilon}(x)] = [\tilde{\varepsilon}(z), \tilde{\varepsilon}(y)] = 1$, whereas $U(\varepsilon)' = [\tilde{\varepsilon}(x), \tilde{\varepsilon}(y)]$, by (3). Therefore, $\tilde{\varepsilon}(z)$ commutes with $U(\varepsilon)'$. As $U(\varepsilon) = \langle \tilde{\varepsilon}(z) \rangle_{z \in P(\Gamma)}$, (2) follows. By (2), $U(\varepsilon)'$ is abelian. Also, given points x, y at distance 3 and elements $v_1, v_2 \in \tilde{\varepsilon}(x)$ and $w \in \tilde{\varepsilon}(y)$, v_2 commutes with $v_1 w v_1^{-1} w^{-1}$ by (2). Hence

$$\begin{aligned}&[v_1, w][v_2, w] = v_1 w v_1^{-1} w^{-1} v_2 w v_2^{-1} w^{-1} = \\ &v_1 v_2 w v_1^{-1} w^{-1} w v_2^{-1} w^{-1} = (v_1 v_2) w (v_1 v_2)^{-1} w^{-1} = [v_1 v_2, w]\end{aligned}$$

Therefore, $[v, w]^p = [v^p, w] = 1$. The first claim of (1) is proved. The rest of (1) follows from the above equality its analogous $[v, w_1][v, w_2] = [v, w_1 w_2]$. ■

Theorem 9.8. *Suppose $q = p$. Then,*

(1) *when $p > 2$, the embedding $\eta : \Gamma \to U$ considered in Lemma* 9.6 *is the abstract hull of ε;*

(2) *when $p = 2$, then $U(\varepsilon) = 2^{1+15}$ or 2^{1+22} according to whether Π is $\mathcal{W}_5(2)$ or $\mathcal{H}_5(2)$.*

Proof. By Proposition 3.6, $U(\varepsilon)/U(\varepsilon)'$ affords the linear hull of ε. Suppose $p > 2$. Then ε_0 is linearly dominant, as remarked in Subsection 9.1. Clearly, the same holds for ε. Furthermore, U/U' can be taken as the codomain of ε, by Lemma 9.5. Hence $U(\varepsilon)/U(\varepsilon)' = U/U'$. Therefore $|U(\varepsilon)| = |U|$ by Lemma 9.7(1). Hence $U(\varepsilon) = U$ and $\tilde{\varepsilon} = \eta$, as claimed in (1).

Suppose $p = 2$. Then, as remarked in Subsection 9.1, the linear hull of ε_0 embeds Γ in $V(15, 2)$ or $V(22, 2)$, according to the type of Γ. So, by an argument similar to the above we obtain that $|U(\varepsilon_0)| = 2^{1+15}$ or 2^{1+22}. However, $U(\varepsilon_0) = U(\varepsilon)$ by Theorem 9.4. The conclusion follows. ■

Corollary 9.9. *Suppose $q = p$. Then,*

(1) *when $p > 2$, $\mathrm{Far}_\Phi(a)$ is the universal cover of $\mathrm{Exp}(\varepsilon)$.*

(2) *when $p = 2$, the universal cover of* $\mathrm{Far}_\Phi(a)$ *is 2-fold or 4-fold, according to whether Φ is of type $F_4(2)$ or ${}^2E_6(2)$.* ■

Note 9.1. A proof of claim (2) of Theorem 9.8 is also given by Ivanov and Shpectorov [18, 3.7.7]. In fact, our proof of Lemma 9.7 is a generalization of their proof.

Problem 3. Does (1) of Theorem 9.8 remain true when K is not a prime field?

9.5 On Case 3 with n=3

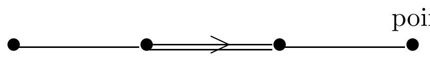
In this subsection, $\Pi = \mathcal{Q}_6(F)$. Accordingly, ε_0 is the spin embedding of Γ in $V(8,K)$. As in the previous subsections, ε is the extension of ε_0 to Γ, $\tilde{\varepsilon}$ and $\tilde{\varepsilon}_0$ are the abstract hulls of ε and ε_0 and $\hat{\varepsilon}$ is the extension of $\tilde{\varepsilon}_0$ to Γ. We put $\Delta := \mathrm{Exp}(\varepsilon)$.

Theorem 9.10. *The geometry Δ is simply connected.*

Proof. The residues of the 3-elements of Δ are expansions of generalized quadrangles embedded in $PG(3,F)$ as symplectic varieties. The conclusion follows from the main Theorem of Cuypers and Van Bon [12]. ■

Corollary 9.11. *The embedding ε is abstractly dominant.* ■

However, Δ is not 2-simply connected, due to the fact that now, differently from what happens in Cases 1 and 2, the residues of the 3-elements of Δ are not simply connected. Actually, as shown by Cuypers and Van Bon [12], Δ is a proper 2-quotient of the geometry $\mathrm{Far}_\Phi(a)$ for Φ a building of type $F_4(K)$ and a a point of Φ, where points are taken as indicated by the following picture:

$(F_4(K))$ 

Explicitly, let A be the elementwise stabilizer in $\mathrm{Aut}(\Phi)$ of the neighbourhood of a in the collinearity graph of Π. Then $\Delta \cong \mathrm{Far}_\Phi(a)/A$. Note also that, denoted by U the unipotent radical of G_a, we have $A \trianglelefteq U$ and the quotient U/A may be regarded as the additive group of an 8-dimensional K-vector space.

As Δ admits a proper 2-cover, ε_0 is not abstractly dominant (Corollary 9.2). In fact, an argument similar to that of the proof of Lemma 9.5 shows that there exists an embedding $\eta_0 : \Sigma \to U$ that properly dominates ε_0 and such that $\mathrm{Exp}(\eta_0)$ is the $\{0,1,2\}$-truncation of $\mathrm{Far}_\Phi(a)$. Denoted by η the extension of η_0 to Γ, we have $\mathrm{Far}_\Phi(a) = \mathrm{Exp}(\eta)$ (but η does not dominate ε, in view of Theorem 9.11). We can summarize the above as follows:

Theorem 9.12. *With A and U as above, let π be the canonical projection of U onto U/A. Then Γ admits an embedding η in U such that $\mathrm{Exp}(\eta) \cong \mathrm{Far}_\Phi(a))$, $\varepsilon = \pi\eta$ and $\varepsilon_0 = \pi\eta_0$. The projection π is not a morphism of embeddings from η to ε (due to the fact that $\eta(X) \cap A \neq 1$ when X is a quad), but it is a morphism from η_0 to ε_0. The geometry $\mathrm{Exp}(\eta)$ is a 2-cover of $\mathrm{Exp}(\varepsilon)$, but not a 3-cover.* ■

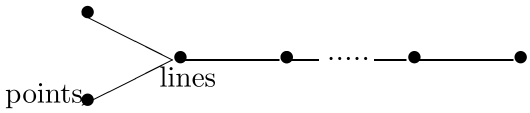
When $\mathrm{ch}(F) = 2$, $\mathcal{Q}_6(F) \cong \mathcal{W}_5(F)$ and Φ is self-dual. In that case, A contains a subgroup $\overline{A} \trianglelefteq G_a$ such that $U/\overline{A}$ is the additive group of $V(15, K)$ (compare Subsection 9.4). In particular, when $F = GF(2)$, the geometry $\mathrm{Far}_\Phi(a)$ admits a 2-fold cover (whence η is not abstractly dominant), we have $|A| = 2^7$ and $|\overline{A}| = 2$.

Problem 4. Is $\mathrm{Far}_\Phi(a)$ simply connected when $F \neq GF(2)$?

Problem 5. Examine Case 4 of Subsection 9.1, but with $n = 3$.

10 Half-spin embeddings

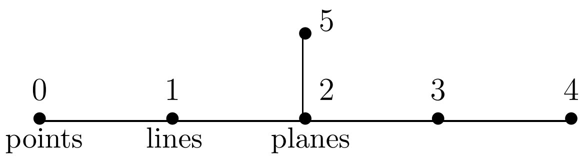
Given a building of type $D_n(K)$, let Σ^+ be one of its two half-spin geometries:



The spin-embedding ε of the dual of $\mathcal{Q}_{2n}(K)$ in $PG(2^n - 1, K)$ (Case 3 of Subsection 9.1) induces a full projective embedding $\varepsilon^+ : \Sigma^+ \to PG(2^{n-1} - 1, K)$, called the *half-spin* embedding (Wells [38]). It is well known that ε^+ is linearly dominant (Wells [38], Shult [32]). When $n = 4$, Σ^+ is isomorphic to the point line system of the polar space $\mathcal{Q}_7^+(K)$ and ε^+ is its (unique) full projective embedding. So, we are back to Section 7 and ε^+ is abstractly dominant, by Theorem 7.7.

Theorem 10.1. *Let $n = 5$. Then ε^+ is abstractly dominant.*

Proof. The additive group of $V(2^{n-1}, K)$ is now isomorphic to the unipotent radical of the stabilizer of A in $\mathrm{Aut}(\Phi)$, for Φ a building of type $E_6(K)$ and A a 4-element of Φ, the types of Φ being chosen as follows:



Accordingly, $\mathrm{Exp}(\varepsilon^+)$ is the point-line-plane system of $\mathrm{Far}_\Phi(A)$. $\mathrm{Far}_\Phi(A)$ is simply connected (see [27]). The residues of elements of $\mathrm{Far}_\Phi(A)$ of type 3, 4 and 5 and of the flags of type $\{3, 5\}$ and $\{3, 4\}$ are simply connected, too. Indeed, they are either affine spaces, or direct sums of an affine space and a geometry of rank 1, or $Af.D_4$-geometries, and it is well known that all of these are simply connected (see [26, Proposition 12.51] for the case of $Af.D_4$). So, an application of [26, Theorem 12.64] is sufficient to see that $\mathrm{Exp}(\varepsilon^+)$, which is the $\{0, 1, 2\}$-truncation of $\mathrm{Far}_\Pi(A)$, is simply connected. The conclusion follows from Corollary 3.4. ■

Problem 6. Describe the abstract hull of ε^+ when $n > 5$ (in particular, for $n = 6$ or 7).

Ackowledgement. The author thanks C. Wiedorn for her careful reading of the earliest draft of this paper.

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