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THE CLASSICAL TRILOGARITHM, ALGEBRAIC K-THEORY OF FIELDS, AND DEDEKIND ZETA FUNCTIONS

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ABSTRACT. In this paper we show how to express the values of $\zeta_F(3)$ for arbitrary number field F in terms of the trilogarithms (D. Zagier's conjecture) and how to relate this result to algebraic K-theory.

1. THE CLASSICAL POLYLOGARITHM FUNCTION

The classical polylogarithm function

$$\mathrm{Li}_p(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^p} (z \in \mathbf{C},\ |z| \leq 1,\ p \in \mathbf{N}) \tag{1.1}$$

during the last 200 years was the subject of much research—see [L]. Using the inductive formula $\mathrm{Li}_p(z) = \int_0^z \mathrm{Li}_{p-1}(t)t^{-1}dt$, $\mathrm{Li}_1(z) = -\log(1-z)$, the p-logarithm can be analytically continued to a multivalued function on $\mathbf{C}\backslash\{0, 1\}$. However, D. Wigner and S. Bloch introduced [B1] the single-valued cousin of the dilogarithm, namely

$$D_2(z) := \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z) \cdot \log|z|. \tag{1.2}$$

Of course, for Li_1 such function is $-\log|z|$. Analogous functions $D_p(z)$ for $p \geq 3$ were introduced in [R] and computed explicitly in [Z]. Let us consider the slightly modified function

$$\mathscr{L}_3(z) := \mathrm{Re}\left[\mathrm{Li}_3(z) - \log|z| \cdot \mathrm{Li}_2(z) + \tfrac{1}{3}\log^2|z| \cdot \mathrm{Li}_1(z)\right]. \tag{1.3}$$

Such modified functions were considered also for all p by D. Zagier, A. A. Beilinson and P. Deligne [Z3, Be1]. $\mathscr{L}_3(z)$ is real-analytic on $CP^1\backslash\{0, 1, \infty\}$ and continuous on CP^1.

Let F be a field. Let P_F^1 be the projective line over F, and let $\mathbf{Z}[P_F^1\backslash 0, 1, \infty]$ be the free abelian group generated by symbols $\{x\}$, where $x \in P_F^1\{0, 1, \infty\}$.

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We may consider $\mathscr{L}_3$ as defining a homomorphism

(1.4) $\mathscr{L}_3\colon \mathbf{Z}[P^1_{\mathbf{C}}\backslash 0, 1, \infty] \to \mathbf{R}, \qquad \mathscr{L}_3\colon \Sigma n_i\{x_i\} \mapsto \Sigma n_i \mathscr{L}_3(x_i).$

We can do the same for any other real-valued function on $P^1_{\mathbf{C}}\backslash\{0, 1, \infty\}$, in particular for D_2.

2. Formula for $\zeta(3)$

Now let F be an arbitrary algebraic number field, d_F the discriminant of F, r_1 and r_2 the number of real and complex places, σ_j all possible embeddings $F \hookrightarrow \mathbf{C}$, $1 \le j \le r_1 + 2r_2$, and $\overline{\sigma_{r_1+k}} = \sigma_{r_1+r_2+k}$. Set $A_{\mathbf{Q}} := A \otimes \mathbf{Q}$. Let us consider the homomorphism

$$\begin{aligned} \Delta\colon \mathbf{Q}[P^1_F\backslash 0, 1, \infty] &\to (\Lambda^2 F^* \otimes F^*)_{\mathbf{Q}}, \\ \Delta\colon \{x\} &\mapsto (1-x)\wedge x \otimes x. \end{aligned} \tag{1.5}$$

Theorem 1. *Let $\zeta_F(s)$ be the Dedekind zeta function of F. Then there exist $y_1, \dots, y_{r_1+r_2} \in \operatorname{Ker}\Delta \subset \mathbf{Q}[P^1_F\backslash 0, 1, \infty]$ such that $\zeta_F(3)$ is equal to $\pi^{3r_2}\cdot|d_F|^{-1/2}$ times the (r_1+r_2)-determinant $\|\mathscr{L}_3(\sigma_j y_i\|\cdot(1 \le j \le r_1 + r_2)$.*

For $s = 2$ a similar result was proved in [Z2]. It also follows directly from results of [Bo, B1, Su]. D. Zagier conjectured that an analogous fact should be valid for all integers $s \ge 3$ [Z3].

To prove Theorem 1 we give an explicit formula expressing the Borel regulator $r_3\colon K_5(\mathbf{C}) \to R$ by $\mathscr{L}_3(z)$, and then use the Borel theorem [Bo]. Below we indicate some ingredients of the proof which are of independent interest.

3. Generic 3-variable functional equation for $\mathscr{L}_3(z)$

The dilogarithm satisfies a remarkable 2-variable functional equation, discovered in the 19th century by W. Spence, N. H. Abel and others [L]. Its version for $D_2(z)$ is as follows. Let $r(x_1, \dots, x_4)$ be the crossratio of a 4-tuple of different points on P^1. For every five different points on P^1 set

$$\begin{aligned} &R_2(x_0, \dots, x_4) := \\ &\sum_{i=0}^{4}(-1)^i[r(x_0, \dots, \hat{x}_i, \dots, x_4)] \in \mathbf{Z}[P^1\backslash 0, 1, \infty]. \end{aligned} \tag{3.1}$$

Then $D_2(R_2(x_0, \dots, x_4)) = 0$ in the sense of formula (1.4). Note that (3.1) depends actually on two variables because of the PGL_2-

invariance of the crossratio. It seems that any other functional equation for $D_2(z)$ can be deduced formally from this one.

It turns out that the analogous functional equation for $\mathscr{L}_3(z)$ corresponds to a special configuration of seven points in the plane. Namely, let x_1, x_2, x_3 be vertices of a triangle in P_F^2 (i.e. these points are not on a line); y_1, y_2, y_3 points on its "sides" $\overline{x_1x_2}$, $\overline{x_2x_3}$, and $\overline{x_3x_1}$, and z a point in generic position (see Figure 1). Further, denote by $(y_1|y_2, y_3, x_3, z)$ the configuration of four points on a line obtained by projection of points y_2, y_3, x_3, z with center at the point y_1. Set

$$\begin{aligned} R_3(x_i, y_i, z) := (1+\tau+\tau^2) \\ \circ [\{r(y_1|y_2, y_3, x_2, z)\} - \{r(y_1|y_2, y_3, x_3, z)\} \\ + \{r(z|x_3, y_3, x_1, y_2)\} + \{r(z|y_3, y_1, x_1, y_2)\} \\ + \{r(z|y_1, x_2, x_1, y_2)\} \\ + \{r(z|x_2, x_3, x_1, y_2)\} - \{r(z|x_3, y_1, x_1, y_2)\}] \\ + \{r(y_1|y_2, y_3, x_2, x_3)\} - 3\{1\} \end{aligned}$$

where $\tau\colon x_i \to x_{i+1}$, $y_i \to y_{i+1}$ (indices modulo 3) (for example, $\tau^2 \circ \{r(y_1|y_2, y_3, x_2, z)\} = \{r(y_3|y_1, y_2, x_1, z)\}$) and, by definition, $\{1\} = \{x\} + \{1-x\} + \{1-x^{-1}\}$ for any $x \in F^*\backslash 1$. As we will see below the choice of x is inessential for our purposes.

Theorem 2. *In the case $F = \mathbf{C}$, $\mathscr{L}_3(R_3(x_i, y_i, z)) = 0$. Note, that $\mathscr{L}_3(\{x\} - \{x^{-1}\}) = 0$ and $\mathscr{L}_3(\{x\}+\{1-x\}+\{1-x^{-1}\}) = \zeta_{\mathbf{Q}}(3)$.*

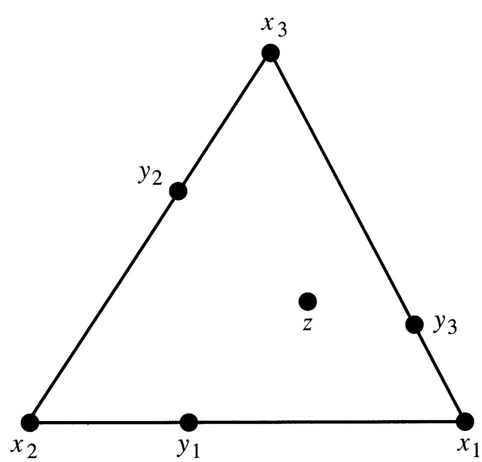

FIGURE 1

A configuration $(x_1, x_2, x_3, y_1, y_2, y_3, z)$ of seven points in P_F^2 depends on three parameters. Consider a specialization of this configuration, when z lies on the line $\overline{x_3y_1}$. It depends on two parameters, and the corresponding functional equation coincides with the classical Spence-Kemmer functional equation for the trilogarithm, discovered by Spence in 1809 [S] and, independently, by E. Kummer in 1840 [K] (see Chapter VI in [L]).

It is also possible to deduce the Spence-Kummer equation formally from Theorem 2 (as a linear combination of relations $\mathscr{L}_3(R_3(x_i, y_i, z)) = 0$). The validity of the inverse statement is an interesting problem.

Conjecture 1. Any functional equation for $\mathscr{L}_3(z)$ can be formally deduced from Theorem 2.

4. Algebraic K-theory of a field

Now let F be an arbitrary field. Set $B_2(F) := \mathbf{Z}[P_F^1\backslash 0, 1, \infty]/R_2$, where R_2 is generated by elements $R_2(x_0, \dots, x_4)$—see (3.1). Then there is the well-known Bloch complex $B_2(F) \xrightarrow{\delta} \Lambda^2 F^*$, where $\delta[x] = (1-x)\wedge x$. (It is not hard to prove that $\delta(R_2) = 0$.). Thanks to Matsumoto, we know that $\mathrm{Coker}\,\delta = K_2(F)([M])$. Using some ideas of S. Bloch [B1], A. Suslin proved that $K_3^{\mathrm{ind}}(F) := \mathrm{Coker}(K_3^M(F) \to K_3(F))$ coincides with $\ker\delta$ modulo torsion [Su].

Note also that $K_1(F) = F^*$ has an interpretation in the same spirit: $F^* = \mathbf{Z}[P_F^1\backslash 0, 1, \infty]/R_1$, where R_1 is generated by expressions $[x]+[y]-[xy]$, reminiscent of the functional equation for $\ln|\cdot|$.

Let us define a complex $\mathbf{Q}(3)_{\mathscr{M}}$ as follows:

$$(4.1) \qquad \mathbf{Q}[P_F^1\backslash 0, 1, \infty]/R_3 \xrightarrow{\delta_1} (B_2(F)\otimes F^*)_{\mathbf{Q}} \xrightarrow{\delta_2} (\Lambda^3 F^*)_{\mathbf{Q}}$$

(the left group placed in degree 1), where $\delta_2[x]\otimes y = (1-x)\wedge x\wedge y$, $\delta_1\{x\} = [x]\otimes x$, and the subgroup R_3 is generated by $\{x\}-\{x^{-1}\}$, $(\{x\}+\{1-x\}+\{1-x^{-1}\}) - (\{y\}+\{1-y\}+\{1-y^{-1}\})$ and $R_3(x_i, y_i, z)$ (see Equation 3.2).

Theorem 2′. $\delta_1(R_3) = 0$ *in* $B_2(F)\otimes F^*$.

Hence the complex $\mathbf{Q}(3)_{\mathscr{M}}$ is well defined. Recall, that $K_n(F) := \pi_n(BGL(F)^+)$, where $BGL(F)^+$ is an H-space. Hence, by the Milnor-Moore theorem [MM] $K_n(F)\otimes\mathbf{Q} = \mathrm{Prim}\,H_n(GL(F), \mathbf{Q})$.

A. Suslin proved [Su2] that $H_n(GL_n(F), \mathbf{Z}) = H_n(GL(F), \mathbf{Z})$. Therefore $K_n(F)\otimes\mathbf{Q} = \operatorname{Prim} H_n(GL_n(F), \mathbf{Q})$. So $\operatorname{Im}(H_n(GL_{n-i}) \to H_n(GL_n))$ gives a canonical filtration $K_n(F)_{\mathbf{Q}} \supset K_n^{(1)}(F)_{\mathbf{Q}} \supset \dots$. Set $K_n^{[m]}(F)_{\mathbf{Q}} := K_n^{(m)}(F)_{\mathbf{Q}}/K_n^{(m+1)}(F)_{\mathbf{Q}}$.

Theorem 3. *There are canonical maps*

$$c_1 : K_5^{[2]}(F)_{\mathbf{Q}} \to H^1(\mathbf{Q}(3)_{\mathcal{M}})$$
$$c_1 : K_4^{[1]}(F)_{\mathbf{Q}} \to H^2(\mathbf{Q}(3)_{\mathcal{M}}).$$

Conjecture 2. c_1 and c_2 are isomorphisms.

Note, that according to [Su2]

$$K_3^{[0]}(F)_{\mathbf{Q}} \backsimeq H^3(\mathbf{Q}(3)_{\mathcal{M}}) \equiv K_3^M(F)_{\mathbf{Q}}.$$

(A. A. Beilinson and S. Lichtenbaum conjectured that there should exist complexes $\mathbf{Q}(j)_{\mathcal{M}}$ computing all $K_n(F)$—see [Be2, Li].)

5. THE GROUP $B_3(F)$

For a **G**-space X, points of $G\backslash X \times \dots \times X$ are called configurations. Let $\mathbf{Z}(C_6(P_F^2))$ be the free abelian group generated by all possible configurations $(l_0, \dots, l_5)$ of 6 points in P_F^2.

Let us define a homomorphism L_3: $\mathbf{Z}[P_F^1\backslash 0, 1, \infty] \to \mathbf{Z}[C_6(P_F^2)]$ as follows: $L_3\{x\} = (x_1, x_2, x_3, y_1, y_2, y_3)$, where $r(y_1|x_1, x_2, y_2, y_3) = x$ (this configuration was described in §3). The (unique) configuration where y_1, y_2, y_3 are on a line will be denoted η_3.

Definition. $B_3(F)$ is the quotient of the group $\mathbf{Z}[C_6(P_F^2)]$ by the following relations

(R1) $(l_0, \dots, l_5) = 0$, if two of the points l_i coincide or four lie on a line.

(R2) (The seven-term relation.) For any seven points $(l_0, \dots, l_6)$ in P_F^2

$$\sum_{i=0}^{6}(-1)^i(l_0, \dots, \widehat{l_i}, \dots, l_6) = 0.$$

(R3) Let $(m_0, \dots, m_5)$ be a configuration of six points in P_F^2, such that $m_2 = \overline{m_0 m_1} \cap \overline{m_3 m_4}$ and m_5 is in generic position—see Fig. 2. Then if $L_3'\{x\} := -L_3\{x\} - 2L_3\{1-x\}$,

$$(m_0, \dots, m_5) = \frac{1}{3}\sum_{i=0}^{4}(-1)^i L_3'\{r(m_5|m_0, \dots, \widehat{m}_i, \dots, m_4)\} + \frac{1}{3}\eta_3.$$

Lemma. *In the group* $B_3(F)$ *we have*

$$(l_0, \dots, l_5) = (-1)^{|\sigma|}(l_{\sigma(0)}, \dots, l_{\sigma(5)}).$$

Remark. The configurations from (R1) are just the unstable ones in the sense of D. Mumford.

Theorem 4. *The homomorphism* $L_3\colon \mathbf{Z}[P_F^1\backslash 0, 1, \infty] \to \mathbf{Z}[C_6(P_F^2)]$ *induces an isomorphism modulo* 6*-torsion.*

$$L_3\colon \mathbf{Z}[P_F^1\backslash 0, 1, \infty]/R_3 \xrightarrow{\sim} B_3(F) \otimes \mathbf{Z}.$$

(It is easy to check using (R2) and (R3) that L_3 is onto; the 7-term relation for a configuration $(x_1, x_2, x_3, y_1, y_2, y_3, z)$ then coincides with $(L_3(R_3(x_i, y_i, z)).)$

Let us denote by M_3 the inverse homomorphism. Then the composition $L_3 \circ M_3\colon B_3(\mathbf{C}) \to \mathbf{Q}[P_{\mathbf{C}}^1\backslash 0, 1, \infty] \to \mathbf{R}$ defines a measurable function on configurations of six points in $\mathbf{C}P^2$, satisfying functional relations (R1) through (R3). So for $x \in P_{\mathbf{C}}^2$, $(L_3 \circ M_3)(x, g_1 x, \dots, g_5 x)$ is a measurable cocycle. Let us prove that its cohomology class lies in $\mathrm{Im}(H_{\mathrm{cts}}^5(GL_3(\mathbf{C}), R) \to H^5(GL_3(\mathbf{C}), R))$, where $H_{\mathrm{cts}}^*(G, R)$ is continuous cohomology.

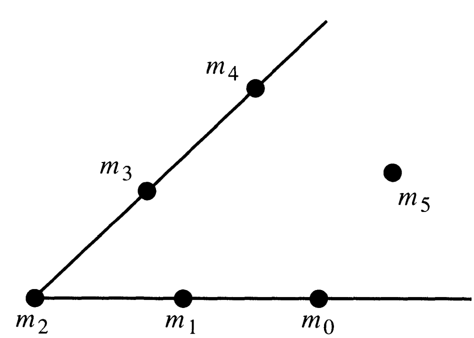

FIGURE 2

Consider the complex

$$\operatorname{Meas} C_{2n-1}(\mathbf{C}P^{n-1}) \xrightarrow{d^*_{2n-1}} \operatorname{Meas} C_{2n}(\mathbf{C}P^{n-1}) \xrightarrow{d^*_{2n}} \operatorname{Meas} C_{2n+1}(\mathbf{C}P^{n-1})$$

where $C_m(\mathbf{C}P^n)$ is the space of all configurations of m points in $\mathbf{C}P^n$, $\operatorname{Meas}(X)$ is the space of all measurable functions on the space X, $d_m: (l_0, \dots, l_m) \mapsto \sum_{i=0}^m (-1)^i (l_0, \dots, \widehat{l_i}, \dots, l_m)$ and d^*_m is the induced map.

Theorem 5. $\operatorname{Ker} d^*_{2n} / \operatorname{Im} d^*_{2n-1}$ *is canonically isomorphic to the indecomposable part of* $H^{2n-1}_{\text{cts}}(GL_n(\mathbf{C}), R)$.

For $n = 2$ this was proved in [B1]. See also closely related work [HM].

Conjecture 3. There exists a canonical element in $\operatorname{Ker} d^*_{2n}$ that can be expressed by classical n-logarithm $\mathscr{L}_n(z)$ and represents the Borel class in $H^{2n-1}_{\text{cts}}(GL_n(\mathbf{C}), R)$.

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