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ALMOST COMMUTING MATRICES AND THE BROWN-DOUGLAS-FILLMORE THEOREM

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The purpose of this note is to announce a constructive proof of the following theorem of Brown, Douglas, and Fillmore [**1**] which yields a quantitative version subject to a certain natural resolvent condition. Complete proofs will appear elsewhere.

THEOREM 1 (BDF). *Let T be an operator on a Hilbert space $\mathcal{H}$ such that $T^*T - TT^*$ is compact, and such that the Fredholm index* $\operatorname{ind}(T-\lambda) = 0$ *whenever this is defined* ($\lambda \notin \sigma_e(T)$). *Then there is a compact operator K such that $T - K$ is normal.*

Our quantitative version yields an estimate of $||K||$ in terms of the homogeneous quantity $||T^*T - TT^*||^{1/2}$ provided the spectrum of T is in a natural quantitative sense close to the essential spectrum $\sigma_e(T)$. Indeed, if N is normal, and $||T - N|| < \varepsilon$, then

$$||(T - \lambda I)^{-1}|| < (\operatorname{dist}(\lambda, \sigma(N)) - \varepsilon)^{-1}.$$

So it is reasonable to assume this inequality when $\operatorname{dist}(\lambda, \sigma(N)) > \varepsilon$.

THEOREM 2. *Given a compact subset X of the plane, there is a continuous positive real-valued function f_X defined on $[0, \infty)$ such that $f_X(0) = 0$ with the following property.*

Let T be essentially normal and satisfy the BDF hypotheses:

(i) $\sigma_e(T) = X$,

(ii) $\operatorname{ind}(T - \lambda I) = 0$ *for all* $\lambda \notin X$.

Furthermore let T satisfy the quantitative hypotheses:

(iii) $||T^*T - TT^*||^{1/2} < \varepsilon$,

(iv) $||(T - \lambda I)^{-1}|| < (\operatorname{dist}(\lambda, X) - \varepsilon)^{-1}$ *if* $\operatorname{dist}(\lambda, X) > \varepsilon$.

Then there is a compact operator K such that $||K|| < f_X(\varepsilon)$ and $T - K$ is a normal operator with spectrum X.

The most important special case in our proof is the annulus $A = \{\lambda \in \mathbf{C}\colon R_1 \le |\lambda| \le R_2\}$ In this case, the result obtained is much stronger and applies to more general operators.

THEOREM 3. *Let T be an operator on a Hilbert space with $\sigma_e(T) = A = \{\lambda \in \mathbf{C}\colon R_1 \le |\lambda| \le R_2\}$. Suppose $||T|| = R_2$ and $||T^{-1}|| = R_1^{-1}$. Then there is an operator K such that*

$$||K|| \le 104||T^*T - TT^*||^{1/2}$$

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and such that $T - K$ is normal with spectrum A. If T is essentially normal, then K may be taken to be compact.

Theorem 3 is obtained by using a polar decomposition analogous to the rectangular decomposition used by the second author in [**2**], where he proves the following absorption theorem, Theorem 4. This construction as described in [**2**] represents the key technical device which allows us to achieve the results of this note.

THEOREM 4. *Let T be a matrix acting on a finite-dimensional Hilbert space $\mathcal{H}$. Then there are normal matrices N and M acting on $\mathcal{H}$ and $\mathcal{H} \oplus \mathcal{H}$ respectively such that $||N|| \leq ||T||$ and*

$$||T \oplus N - M|| \leq 75||T^*T - TT^*||^{1/2}.$$

Let us briefly sketch the scheme of the proof of Theorem 4. We will avoid technical difficulties that would arise in getting the estimate of the right order. Write $T = A + iB$ as a sum of its real and imaginary parts. Let $\varepsilon^2 = ||T^*T - TT^*|| = 2||AB - BA||$. Use the spectral decomposition of A to split the space into a direct sum of spectral subspaces for intervals of length ε. With respect to this decomposition, A is "block diagonal". Since $||AB - BA|| \leq \varepsilon^2/2$, it follows that the matrix entries B_{ij} of B satisfy $||B_{ij}|| = O(\varepsilon)$ for $|i - j| \geq 2$. So after a $O(\varepsilon)$ perturbation, B is "tridiagonal". Let $R = [\varepsilon^{-1/2}]$ and let B' be the tridiagonal matrix obtained from B by setting the $(kR, kR+1)$ and $(kR+1, kR)$ entries equal to zero. This B' is block diagonal with respect to this coarser decomposition corresponding to spectral subspaces of A for intervals of length $R\varepsilon \doteqdot \varepsilon^{1/2}$. Let A' be the block diagonal matrix which is scalar on each big block, with value equal to the midpoint of the corresponding interval. So $||A' - A|| \doteqdot \varepsilon^{1/2}/2$. The normal matrix $N = A' + iB'$ works. The point is that $T \oplus N$ is close to $(A' \oplus A') + i(B \oplus B')$. Since B and B' agree on long sequences of blocks it is possible to think of them "locally" as looking like $B \otimes I_2$, which is a tridiagonal matrix with entries

$$\begin{pmatrix} B_{ij} & 0 \\ 0 & B_{ij} \end{pmatrix} = B_{ij} \otimes I_2,$$

where I_2 is the 2×2 identity matrix. One writes down a projection which is a block diagonal operator with respect to the original decomposition, of the form $\sum_{j=-R}^{R} \oplus I \otimes P_j$ where P_j slowly varies from $\left(\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and back to $\left(\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right)$. A computation shows that this almost commutes with $T \oplus N$ because of the zero off-diagonal entries of B' at every Rth block. In this way $B \oplus B'$ is approximated by a block diagonal operator B_0. The block projections commute with $A' \oplus A'$ and are supported on two adjacent big blocks. So $A' \oplus A'$ is close to a selfadjoint operator A_0 which is scalar on each of the block projections of B_0. So $A_0 + iB_0$ is normal, and close to $T \oplus N$.

Now let us show how Theorem 3 is applied to the proof of Theorem 2. Given an essentially normal operator T with zero index data, it is always possible to approximate T by an operator unitarily equivalent to $T \oplus N$ where N is normal with $\sigma(N) = \sigma_e(N) = \sigma_e(T)$. This normal can in turn be approximated by a normal operator N_1 with a bigger spectrum which is topologically nice (finitely connected, smooth boundary). A conformal mapping is used to straighten this set out. Then a small perturbation splits this operator into a direct sum of operators $\sum_{i=1}^{n} \oplus T_i$ so that $\sigma_e(T_i)$ is conformally equivalent to the annulus. By Theorem 3, each T_i is normal plus compact. Hence T is the *limit* of operators which are normal plus compact.

An operator T is called *quasidiagonal* if there is an increasing sequence P_n of finite rank projections with $s - \lim P_n = I$ such that

$$\lim_{n\to\infty} ||P_nT - TP_n|| = 0.$$

Such operators have a small compact perturbation $T - K$ which is of the form $\sum_{n\geq 1} \oplus T_n$ where each T_n acts on a finite-dimensional space. It is a consequence of the spectral theorem that every normal operator is quasidiagonal. The set of quasidiagonal operators is norm closed and invariant under compact perturbations. Hence our essentially normal operator T with zero index data is quasidiagonal.

This brings us to the matrix problem. Since $\sum_{n\geq 1} \oplus T_n$ is essentially normal, it follows that

$$\lim_{n\to\infty} ||[T_n^*, T_n]|| = 0.$$

The following problem is still open.

PROBLEM 1. *Let T be a matrix of norm one such that $||T^*T - TT^*||$ is small. Is T close to a normal matrix?*

More precisely, given $\varepsilon > 0$, is there a $\delta > 0$ so that $||T^*T - TT^*|| < \delta$ implies that $\operatorname{dist}(T, \text{Normals}) < \varepsilon$? This problem has been considered by several authors, in particular the first author of this note, and the interested reader is referred to [**2**] for further information.

Theorems 3 and 4 provide a method for circumventing Problem 1 while still obtaining a result good enough for BDF because of the abundance of normal summands available. Because of the existence of isolated eigenvalues, a solution to Problem 1 is required if there is to be an "ideal" quantitative version. Theorem 5 below is a generalization of Theorems 3 and 4, which correspond to the annulus and disc cases. Let X be a compact subset of the plane, and let $\operatorname{Nor}(X)$ be the set of normal matrices with spectrum contained in X. For $t \geq 0$, let $S_t(X)$ denote the set of matrices T such that

(i) $||T^*T - TT^*||^{1/2} \leq t$, and
(ii) $||(T - \lambda I)^{-1}|| \leq (\operatorname{dist}(\lambda, X) - t)^{-1}$ for λ such that $d(\lambda, X) > t$.

THEOREM 5. *There is a continuous function f_X defined on $[0, \infty)$ with $f_X(0) = 0$ so that if T belongs to $S_t(X)$, there are normal matrices N and M in* $\operatorname{Nor}(X)$ *such that $||T \oplus N - M|| < f_X(t)$.*

Theorems 1 and 2 now follow. Indeed, let T be an essentially normal operator with zero index data. We have shown that T is quasidiagonal. So we see $T - K \cong \sum_{n\geq 1} \oplus T_n \oplus N$, where N is a diagonal normal operator and $\sigma(N) = \sigma_e(T) = X$. The finite-dimensional operator T_n belongs to $S_{t_n}(X)$ and $\lim_{n\to\infty} t_n = 0$. If T satisfies the hypotheses of Theorem 2, then one also can arrange that $\sup_{n\geq 1} t_n \leq \varepsilon$. Using Theorem 5, one attaches to each T_n a summand N_n of N so that there is a normal matrix M_n with $\|T_n \oplus N_n - M_n\| \leq t_n$. So

$$T - K \cong \sum_{n\geq 1} \oplus (T_n \oplus N_n) \oplus N$$

which differs from the normal operator $\sum_{n\geq 1} \oplus M_n \oplus N$ by a compact operator of norm at most $f_X(\sup_{n\geq 1} t_n)$.

There are many questions left open for consideration. One of the most pertinent problems is the following.

PROBLEM 2. *Is there a universal constant C so that if T is a matrix with $\|T^*T - TT^*\|^{1/2} = t$, then there are normal matrices N and M so that the eigenvalues of N belong to $\{\lambda \in \mathbf{C}\colon \|(T-\lambda I)^{-1}\| > t^{-1}\}$ and $\|T \oplus N - M\| \leq Ct$?*

A positive solution would yield a stronger version of Theorem 2.

REFERENCES

1. L. G. Brown, R. G. Douglas, and P. A. Fillmore, *Unitary equivalence modulo the compact operators and extensions of C^*-algebras*, Proc. Conf. Operator Theory, Lecture Notes in Math., vol. 345, Springer-Verlag, Berlin and New York, 1973, pp. 58–128.

2. K. R. Davidson, *Almost commuting Hermitian matries*, Math. Scand. **56** (1985), 222–240.

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