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NONCLASSICAL EIGENVALUE ASYMPTOTICS FOR OPERATORS OF SCHRÖDINGER TYPE

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We consider operators in the form $A = -\nabla \cdot \rho\nabla + V(x)$ on $\mathbf{R}^n$, where metric $\rho = (\rho_{ij}(x)) \geq 0$ and potential $V(x) \geq 0$. The classical Weyl principle for asymptotic distribution of large eigenvalues of A states that the counting function

$$N(\lambda) = \#\{\lambda_j \leq \lambda\} \sim \operatorname{Vol}\{(x;\xi)|\ \rho\xi\cdot\xi + V(x) \leq \lambda\} \quad \text{as } \lambda \to \infty.$$

(See for instance [**Gu**].) Integrating out variable ξ we can rewrite it as

$$N(\lambda) \sim \frac{\omega_n}{(2\pi)^n} \int (\lambda - V)_+^{n/2} \frac{dx}{\sqrt{\det\rho}}. \tag{1}$$

If potential V and metric ρ are assumed to be homogeneous in x, $V(x) = |x|^\alpha V(x')$; $\rho_{ij}(x) = |x|^\beta \rho_{ij}(x')$, $x' = x/|x|$, then (1) reduces to

$$N(\lambda) \sim C\lambda^{[n/2+(1-\beta/2)n/\alpha]} \int V^{-(n/\alpha)(1-\beta/2)} \frac{dS}{\sqrt{\det\rho}}; \tag{2}$$

integration over the unit sphere S with constant

$$C = \frac{\omega_n}{(2\pi)^n\alpha} B\left(\frac{n}{2} + 1; \frac{n}{\alpha}(1-\beta/2)\right),$$

which depends on the volume ω_n of the unit sphere in $\mathbf{R}^n$ and the beta function.

Assuming $\beta < 2$ we see that integral (2) becomes divergent if $V(x')$ vanishes to a sufficiently high order. The simplest such potential is $V(x,y) = |x|^\alpha|y|^\beta$ on $\mathbf{R}^n + \mathbf{R}^m$.

The Weyl (volume counting) principle, when applied to the corresponding Schrödinger operator $-\Delta + V(x)$, fails to predict discrete spectrum below any energy level $\lambda > 0$. However, as was shown by D. Robert [**Ro**] and B. Simon [**Si**], A has purely discrete spectrum $\{\lambda_j\} \to +\infty$ (for qualitative explanation of this phenomenon see [**Fe**]). Moreover, the "nonclassical" asymptotics of $N(\lambda)$ was derived for such A.

Recently M. Solomyak [**So**] studied a general class of Schrödinger operators $-\Delta + V(x)$ with homogeneous potentials V subject to the following constraint:

(A) zeros of V, $\{x\colon V(x) = 0\}$ form a smooth cone Σ in $\mathbf{R}^n$ of dimension m, and V vanishes on Σ "uniformly" to order $b > 0$.

Introducing variables $x \in \Sigma$ and $y \in N_x$ (the normal to Σ at $\{x\}$), hypothesis (A) means that there exists

$$\lim_{t\to 0} t^{-b} V(x + ty) = V_0(x,y).$$

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It is easy to see that $V_0(x,y)$ has mixed homogeneity

$$V_0(x,y) = |x|^a|y|^b V_0(x',y'); \quad a+b=\alpha \tag{3}$$

and V_0 approximates V in a small conical neighborhood Σ_ε of Σ:

$$\Sigma_\varepsilon = \{x+y|\ x\in\Sigma; |y|<\varepsilon|x|\}.$$

Under hypothesis (A) M. Solomyak [**So**] derived asymptotics of $N(\lambda)$ for such operators $A = -\Delta + V(x)$ in terms of eigenvalues $\{\lambda_j(x)\}_1^\infty$ of an auxiliary family of Schrödinger operators $\{L(x) = -\Delta_y + V_0(x,y)\}_{x\in\Sigma}$. Namely,

$$N(\lambda) \sim C\lambda^{\frac{m}{2}(1+\frac{2+b}{a})} \int_{\Sigma'} \sum_1^\infty \lambda_j(x')^{-m(2+b)/2a}\, dS, \tag{4}$$

the integral is over $\Sigma' = \Sigma\cap S$ (unit sphere).

Notice that each operator $L(x)$ has "classical type," so Weyl's principle (2) applies to $\{\lambda_j(x)\}_1^\infty$,

$$\#\{\lambda_j(x)\le\lambda\} \sim c(x)\lambda^{(n-m)(1/2+1/b)}. \tag{5}$$

Let us also observe that a polynomial asymptotics of $N(x)\sim c\lambda^p$ implies convergence of the series

$$\sum_1^\infty \lambda_j^{-q} < \infty, \text{ with any } q>p.$$

Hence by (5) the sum in (4) converges provided

$$q = m(2+b)/2a > p = (n-m)(1/2+1/b). \tag{6}$$

Condition (6) is sufficient for validity of (4). In the critical case $q=p$ an additional $\log\lambda$ factor appears in (4).

The method of [**So**] was based on the variational formulation of the problem and certain eigenvalue estimates for Schrödinger operators in conical regions obtained in [**Ros**].

In the present paper we shall outline a different approach based on pseudodifferential calculus with operator-valued symbols in the spirit of [**Ro**]. This method allows us to recover Solomyak's result (4) and to extend it in various directions, including operators of the form $-\nabla\cdot\rho\nabla + V(x)$.

We propose the following principle, which governs nonclassical asymptotics: the main contribution to $N(\lambda)$ comes from the degeneracy set Σ (critical set) of V.

According to this principle we want to "localize" A to a small (conical) neighborhood of Σ. Precisely, let us introduce the "model" operator

$$A_0 = -\Delta_\Sigma + L(x) = -\Delta_\Sigma + [-\Delta_N - 2\nabla_x\cdot\rho'\nabla_y + V_0(x,y)] \tag{7}$$

on the manifold $\mathcal{N}(\Sigma) = \bigcup_{x\in\Sigma} N_x$, normal bundle to Σ, where Δ_Σ, Δ_N are the Laplace-Beltrami operators on Σ and the normal space, $N = N_x$, with respect to the metrics induced by ρ_{ij}, and ρ' is the "off diagonal" part of ρ.

Writing $A = -\nabla\cdot\rho\nabla + V$ in normal coordinates (x,y) one can show that $A = A_0+$"small perturbation" in a conical neighborhood Σ_ε of Σ. So we expect $N(\lambda;A)\sim N(\lambda;A_0)$, as $\lambda\to\infty$.

To study the eigenvalue distribution one usually works with certain integral "transforms" of $N(\lambda)$, like $\operatorname{tr} e^{-tA} = \int^{+\infty} e^{-\lambda t}\, dN(\lambda)$ or $\operatorname{tr}(\varsigma + A)^{-l} = \int^{+\infty} (\varsigma + \lambda)^{-l}\, dN(\lambda)$.

We prefer to work with the latter. Once the asymptotics

$$\operatorname{tr}(\varsigma + A)^{-l} \sim c_0 \varsigma^{-l+p} \quad \text{as } \varsigma \to \infty \tag{8}$$

is established for $\operatorname{tr} R^l_S$ one can go back to the asymptotics of $N(\lambda) \sim c\lambda^p$, as $\lambda \to \infty$, by the Tauberian Theorem of M. V. Keldysh (see [**Ro**]). The relation between the two constants is $c = c_0/pB(p; l-p)$.

So we need to establish (8).

Operator A can be thought of as a differential operator on Σ with operator-valued symbol $\sum g^{ij}\xi_i\xi_j + L(x)$, where metric $g = \rho_\Sigma - \rho'^* \rho_N^{-1} \rho'$ on Σ is constructed from the tangent ρ_Σ and normal ρ_N components of ρ. Then the parametrix (approximate inverse) of $(\varsigma + A_0)^{-l}$ can be constructed as an operator-valued ΨDO $K = K^{(l)}_\varsigma$ with symbol

$$\sigma_K = \left[\varsigma + \sum g^{ij}\xi_i\xi_j + L(x)\right]^{-l}.$$

According to our principle we want to localize kernels $R^l_\varsigma = (\varsigma + A)^{-l}$; $\tilde{R}^l = (\varsigma + A_0)^{-l}$ and $K^{(l)}_\varsigma$ to a small conical neighborhood Σ_ε of Σ. Let us introduce a cut-off function

$$\chi_\varepsilon = \begin{cases} 1 & \text{on } \Sigma_\varepsilon, \\ 0 & \text{outside,} \end{cases}$$

and define an orthogonal projection $P_\varepsilon u = \chi_\varepsilon u$ from $L^2(\mathbf{R}^n)$ onto $L^2(\Sigma_\varepsilon)$.

The following lemma plays the central role in the localization procedure.

LEMMA. *All traces below are equivalent as* $\varsigma \to \infty$.
(i) $\operatorname{tr}(\varsigma + A)^{-l} \sim \operatorname{tr} P(\varsigma + A)^{-l}P$,
(ii) $\operatorname{tr}(\varsigma + A_0)^{-l} \sim \operatorname{tr} P(\varsigma + A_0)^{-l}P$,
(iii) $\operatorname{tr} K^{(l)}_\varsigma \sim \operatorname{tr} PK^{(l)}_\varsigma P$,
(iv) *traces of "truncated" operators* : $P(\varsigma + A)^{-l}P$, $P(\varsigma + A_0)^{-l}P$, *and* $PK^{(l)}_\varsigma P$ *are all equivalent.*

From the lemma follows

$$\operatorname{tr}(\varsigma + A)^{-l} \sim \operatorname{tr} K^{(l)} \quad \text{as } \varsigma \to \infty. \tag{9}$$

Now it remains to compute the trace of an operator-valued ψDO $K^{(l)}_\varsigma$

$$\operatorname{tr} K^{(l)}_\varsigma = \iint \sum_{k=1}^{\infty} \left[\varsigma + \sum g^{ij}\xi_i\xi_j + \lambda_k(x)\right]^{-l} d\xi\, dx. \tag{10}$$

Integrating out variables ξ, using homogeneity of $\lambda_j(x)$ and $\rho(x)$, and introducing "polar coordinates" on Σ to reduce integration over the cone Σ to a subset $\Sigma' = \Sigma \cap S$, we get

$$\operatorname{tr} K^{(l)}_\varsigma = C_0 \varsigma^{-l+m(1/2+\theta)} \int_\Sigma \sum_1^\infty \lambda_j(x')^{-m\theta} \frac{dx'}{\sqrt{\det g^{ij}(x')}} \tag{11}$$

with constants

$$(12)\quad s=\frac{\beta b+2a}{2+b};\quad \theta=\frac{1}{s}(1-\beta/2);\quad C_0=\int_0^\infty r^{m(1-\beta/2)}(1-r^s)^{m/2-l}\,dr.$$

Remembering that $\{\lambda_j(x')\}$ obey the classical asymptotics (5) with exponent $p=(n-m)(2+b)/2b$, we obtain a sufficient condition of convergence of series (11)

$$(13)\quad m\theta=\frac{m}{s}(1-\beta/2)>p=\frac{(n-m)(2+b)}{2b}\quad\text{or}\quad\frac{m(2-\beta)}{b+2a}>\frac{n-m}{b}.$$

Thus we have established the following

THEOREM. *If operator $A=-\nabla\cdot\rho\nabla+V$ with homogeneous potential $V(x)=|x|^\alpha V(x')\geq 0$ and nondegenerate metric $\rho_{ij}(x)=|x|^\beta\rho_{ij}(x')>0$ satisfies hypothesis (A), then spectral function $N(\lambda)$ of A admits the nonclassical asymptotics*

$$(14)\qquad N(\lambda)\sim C\lambda^{m(1/2+\theta)}\int_{\Sigma'}\sum_1^\infty\lambda_j(x')^{-m\theta}\frac{dx'}{\sqrt{\det g^{ij}(x')}},$$

provided sufficient condition (13) *holds. The metric (g^{ij}) on Σ is obtained from components of metric ρ.*

REMARKS. (i) Formula (14) includes both the classical formula (2) with $\beta=0$ and $s=a$ (i.e., $b=0$) and all previously studied nonclassical asymptotics [**Ro**, **Si**, **So**] (the latter corresponds to $\beta=0$).

(ii) In the critical case (equality $m\theta=p$ in (13)) an additional $\log\lambda$ factor appears in (16). The argument requires some modification: Before passing to the limit in the sum $\sum_1^\infty\lambda_j^{-m\theta}$ and integration over Σ one has to "localize" $K_\varsigma^{(l)}$ to a compact region in Σ.

We shall illustrate our theorem and conditions by the following

EXAMPLE. Take scalar metric $(\rho_{ij})=\rho=(t^2+|x|^2)^{\beta/2}I_{n\times n}$ and potential $V=(t^2-|x|^2)^{\beta/2}$ in the space $\mathbf{R}^n=\{(t,x)\colon t\in\mathbf{R};\ x\in\mathbf{R}^{n-1}\}$. The degeneracy set of V is the standard cone $\Sigma=\{(t,x)\colon t=\pm|x|\}$ in $\mathbf{R}^n$.

Direct calculation shows: $a=b=\alpha/2$ and $V_0(x,y)=|x|^{\alpha/2}|y|^{\alpha/2}$.

Condition (15) for convergence of the series of eigenvalues $\sum_j\lambda_j^{-(n-1)\theta}$ of the operator $L(x)=-d^2/dy^2+|y|^{\alpha/2}$ on $\mathbf{R}$ becomes

$$\frac{\beta+2}{2-\beta}<n-1\quad\text{or}\quad\beta<\frac{2(n-2)}{n},$$

and the eigenvalue asymptotics takes a form

$$N(\lambda)\sim C\lambda^{(n-1)(1/2+\theta)}\sum_1^\infty\lambda_j^{-(n-1)\theta}\quad\text{with }\theta=\frac{(4+\alpha)(2-\beta)}{2\alpha(\beta+2)}.$$

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