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COMPACT RIEMANNIAN MANIFOLDS WITH POSITIVE CURVATURE OPERATORS

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The Riemann-Christoffel curvature tensor R of a Riemannian manifold M determines a *curvature operator*

$$\mathcal{R}: \Lambda^2 T_p M \to \Lambda^2 T_p M,$$

where $\Lambda^2 T_p M$ is the second exterior power of the tangent space $T_p M$ to M at p, by the explicit formula

$$\langle \mathcal{R}(x \wedge y), z \wedge w \rangle = \langle R(x, y)w, z \rangle.$$

M is said to have *positive curvature operators* if the eigenvalues of $\mathcal{R}$ are positive at each point $p \in M$. Meyer used the theory of harmonic forms to prove that a compact oriented n-dimensional Riemannian manifold with positive curvature operators must have the real homology of an n-dimensional sphere [**GM**, Proposition 2.9]. Using the theory of minimal two-spheres, we will outline a proof of the following stronger result.

THEOREM 1. *Let M be a compact simply connected n-dimensional Riemannian manifold with positive curvature operators, where $n \geq 4$. Then M is homeomorphic to a sphere.*

Theorem 1 is actually a consequence of another theorem which makes a weaker hypothesis on the curvature tensor. To describe this hypothesis, we extend the Riemannian metric $\langle\ ,\ \rangle$ in two ways to the complexified tangent space $T_p M \otimes \mathbf{C}$: as a complex symmetric bilinear form $(\ ,\ )$ and as a Hermitian inner product $\langle\langle\ ,\ \rangle\rangle$. Similarly, we extend the metric in two ways to $\Lambda^2 T_p M \otimes \mathbf{C}$. An element $z \in T_p M \otimes \mathbf{C}$ is said to be *isotropic* if $(z, z) = 0$. A complex linear subspace $V \subseteq T_p M \otimes \mathbf{C}$ is *totally isotropic* if $z \in V \Rightarrow (z, z) = 0$.

Finally, we extend the curvature operator $\mathcal{R}$ to a complex linear map $\mathcal{R}: \Lambda^2 T_p M \otimes \mathbf{C} \to \Lambda^2 T_p M \otimes \mathbf{C}$.

DEFINITION. The curvature operator $\mathcal{R}$ is *positive on complex totally isotropic two-planes* if whenever $\{z, w\}$ is a basis for a totally isotropic subspace of $T_p M \otimes \mathbf{C}$ of complex dimension two,

$$\langle\langle \mathcal{R}(z \wedge w), z \wedge w \rangle\rangle > 0.$$

(Note that M has positive sectional curvatures if and only if its curvature operator $\mathcal{R}$ is positive on *real* two-planes.)

By means of a purely algebraic argument, it is possible to prove that if the sectional curvatures $K(\sigma)$ of a Riemannian manifold M of dim ≥ 4 satisfy the inequality $1/4 < K(\sigma) \leq 1$, then the curvature operator of M is positive

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on complex totally isotropic two-planes. Hence the following theorem implies not only Theorem 1, but also the Rauch-Berger-Klingenberg sphere theorem for manifolds of dim ≥ 4 [**GKM**, §7.8].

THEOREM 2. *Let M be a compact simply connected n-dimensional Riemannian manifold whose curvature operator is positive on complex totally isotropic two-planes, where $n \geq 4$. Then M is homeomorphic to a sphere.*

SKETCH OF PROOF OF THEOREM 2. Let $S^2 = \mathbf{C} \cup \{\infty\}$ be the Riemann sphere with the standard complex coordinate $z = x + iy$. If $f: S^2 \to M$ is a conformal branched minimal immersion, and V is a C^∞ section of f^*TM, then the second derivative of the energy in the direction of the variation field V is given by the index form

(1)
$$I(V,V) = \int_{S^2} \{|\nabla_{\partial/\partial x} V|^2 + |\nabla_{\partial/\partial y} V|^2 - \langle R(V, \partial f/\partial x)\partial f/\partial x, V\rangle - \langle R(V, \partial f/\partial y)\partial f/\partial y, V\rangle\}\, dx \wedge dy.$$

It is convenient to extend this index form to a Hermitian symmetric form on C^∞ sections of $f^*TM \otimes \mathbf{C}$. If $W = U + iV$, where U and V are smooth sections of f^*TM, then integration by parts (as in [**M**]) yields the formula

(2)
$$\begin{aligned} I(W,W) &= I(U,U) + I(V,V) \\ &= 4\int_{S^2} \{\|\nabla_{\partial/\partial \bar z} W\|^2 - \langle\langle \mathcal{R}(W \wedge \partial f/\partial z), W \wedge \partial f/\partial z\rangle\rangle\}\, dx \wedge dy. \end{aligned}$$

In this formula, $\partial f/\partial z$ is the section of $f^*TM \otimes \mathbf{C}$ defined by

$$(\partial f/\partial z)(p) = (1/2)(f_{*_p}(\partial/\partial x|_p) - i f_{*_p}(\partial/\partial y|_p)), \quad \text{for } p \in S^2.$$

$f^*TM \otimes \mathbf{C}$ can be made into a holomorphic vector bundle over S^2 in a unique fashion so that the local holomorphic sections of $f^*TM \otimes \mathbf{C}$ are exactly the sections annihilated by $\nabla_{\partial/\partial \bar z}$. When this is done, the fact that f is conformal and harmonic implies that $\partial f/\partial z$ is an isotropic holomorphic section of $f^*TM \otimes \mathbf{C}$. A theorem of Grothendieck [**G**] implies that $f^*TM \otimes \mathbf{C}$ can be decomposed into a direct sum of holomorphic line bundles, $f^*TM \otimes \mathbf{C} = L_1 \oplus L_2 \oplus \cdots \oplus L_n$, where

$$c_1(L_1) \geq c_1(L_2) \geq \cdots \geq c_1(L_n), \qquad c_1(L_{n-i}) = -c_1(L_i).$$

(Here $c_1(L_i)$ denotes the first Chern class of L_i evaluated on the fundamental cycle of S^2.) This direct sum decomposition allows us to give a lower bound on the dimension of the space of isotropic holomorphic sections of $f^*TM \otimes \mathbf{C}$, and this bound, together with formula (2), can be used to establish

PROPOSITION. *If $f: S^2 \to M$ is a nonconstant conformal branched minimal immersion into a Riemannian manifold whose curvature operator is positive on complex totally isotropic two-planes, then the index form* (1) *at f has index $\geq (n/2) - (3/2)$.*

(By the *index* of a symmetric bilinear form, we mean the dimension of a maximal linear subspace of the domain on which the form is negative definite.)

Now we utilize the α-energy of Sacks and Uhlenbeck [**SU**], the real-valued C^2 function on the Banach manifold $L_1^{2\alpha}(S^2, M)$, where α is slightly greater than 1, defined by

$$E_\alpha(f) = \int_{S^2} (1 + |df|^2)^\alpha \, d\mu.$$

Here S^2 is given the metric of constant curvature having volume one, $d\mu$ is the area element with respect to this metric, and $|df|^2$ is the energy density. We regard M as isometrically imbedded in an ambient Euclidean space E^N, and set

$$T_f L_1^{2\alpha}(S^2, M) = \{V: S^2 \to E^N \mid V(p) \in T_{f(p)}M, \quad \text{for all } p \in S^2\}.$$

To any critical point f for E_α is associated its Hessian, a continuous symmetric bilinear form

$$d^2 E_\alpha(f): T_f L_1^{2\alpha}(S^2, M) \times T_f L_1^{2\alpha}(S^2, M) \to \mathbf{R}.$$

LEMMA. *Let k be the least integer, $2 \leq k \leq n$, such that $\pi_k(M) \neq 0$. Then there is a nonconstant critical point f for E_α such that the Hessian of E_α at f has index $\leq k - 2$.*

Indeed, if E_α did not have any nonconstant critical points of index $\leq k-2$, it could be approximated by a function whose nonconstant critical points were weakly nondegenerate (in the sense of Uhlenbeck [**U**, p. 432]) and of index $\geq k - 1$. Then Morse theory on Banach manifolds [**U, T**] would imply vanishing of the relative homotopy group

$$\pi_{k-2}(L_1^{2\alpha}(S^2, M), \hat{M}) = 0,$$

where $\hat{M}$ is the subspace of constant maps from S^2 to M. This would contradict $\pi_k(M) \neq 0$.

By the lemma, we can choose a sequence of nonconstant critical points $f_{\alpha(i)}$ for $E_{\alpha(i)}$ of index $\leq k - 2$, with $\alpha(i) \downarrow 1$. By [**SU**], we can assume that $E_\alpha(f_\alpha) \leq (1 + B^2)^\alpha$ and energy $(f_\alpha) \geq \varepsilon$, where B and ε are positive constants independent of α. After passing to a subsequence, we can arrange that the $f_{\alpha(i)}$'s will C^1-converge on S^2 minus a finite number of points to a conformal branched minimal sphere [**SU**, Theorem 4.4]. If the limiting sphere is nonconstant, it can be shown that its index form has index $\leq k-2$. (We can neglect the finite number of points at which convergence fails by an argument of Gulliver and Lawson [**GL**, Proposition 1.9].)

If the limiting sphere is constant, then a nontrivial branched conformal minimal sphere must "bubble off" as $\alpha \to 1$ [**SU**, Theorem 4.6]. In this case, a nontrivial bubbled-off sphere must have index $\leq k - 2$.

The proposition now implies that $k-2 \geq (n/2) - (3/2)$. Hence $\pi_i(M) = 0$, for $1 \leq i \leq n/2$. It thus follows from Poincaré duality that M is a homotopy sphere, and by the resolutions of the generalized Poincaré conjecture when $n \geq 4$, M must be homeomorphic to a sphere.

More details will appear in a subsequent article.

ADDED IN PROOF. The author has recently been informed that Micallef has independently obtained results similar to Theorems 1 and 2.

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