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AN ALMOST-ORTHOGONALITY PRINCIPLE WITH APPLICATIONS TO MAXIMAL FUNCTIONS ASSOCIATED TO CONVEX BODIES

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1. Introduction. Let B be a convex body in R^n, normalised to have volume one. Let M be the centred Hardy-Littlewood maximal function defined with respect to B, i.e.

$$Mf(x) = \sup_t t^{-n} \int_{tB} |f(x-y)|\, dy.$$

Let $\tilde{M}$ be the lacunary maximal operator,

$$\tilde{M}f(x) = \sup_k 2^{-kn} \int_{2^k B} |f(x-y)|\, dy.$$

Considerable interest has recently been shown in the behaviour of these operators for large n, see [**1**, **2**, **8**, **9**, **10**]. When B is the ball, Stein has shown [**8**] that M is bounded on $L^p(R^n)$, $1 < p \leq \infty$, with a constant C_p depending only on p, and not on n; Stein and Strömberg [**10**] have shown that for p larger than 1, the L^p operator norm of M is at most linear in the dimension. More recently Bourgain has proved that the L^2 operator norm of M is bounded by an absolute constant independent of the body and the dimension [**1**]. It is the purpose of this note to extend this result to $p > 3/2$, and to all $p > 1$ if we instead consider $\tilde{M}$.

THEOREM 1. (i) *Let $p > 3/2$. Then there exists a constant C_p, depending only on p and not on B or n, such that* $\|Mf\|_p \leq C_p \|f\|_p$.

(ii) *Let $p > 1$. Then there exists a constant D_p, depending only on p and not on B or n, such that* $\|\tilde{M}f\|_p \leq D_p \|f\|_p$.

It has recently been brought to the author's attention that part (i) of the theorem has been proved by Bourgain[2] in the special case that B is the cube [**2**]. Here we show that Theorem 1 in fact follows from Bourgain's previous analysis together with a general almost-orthogonality principle for maximal functions, Theorem 2. A weaker version of this principle appears in [**6**], where it is also applied to various operators including maximal functions and Hilbert transforms along curves. A similar principle due to Michael Christ appears in [**4**].

Full details of the proofs, together with further applications, will appear elsewhere.

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[2]NOTE ADDED IN PROOF. Theorem 1 has been proved in full independently by J. Bourgain.

2. An almost-orthogonality principle. Let T_{jv}, $j \in Z$, $v \in S$, be a family of linear operators (multiplier operators in our application). Here S is any indexing set. Suppose there are linear operators R_j ($j \in Z$) such that $\sum R_j = I$. Consider the maximal operator $T_* f(x) = \sup_{jv} |T_{jv} f(x)|$.

DEFINITION. (i) T_* is weakly bounded (with respect to R_j) on L^q if

$$\sup_k \left\| \sup_{jv} |T_{jv} R_{j+k} f| \right\|_q \leq C \|f\|_q.$$

(ii) T_* is strongly bounded on L^q if for some sequence a_k satisfying $\sum a_k^t < \infty$, $0 < t \leq 1$,

$$\left\| \sup_{jv} |T_{jv} R_{j+k} f| \right\|_q \leq a_k \|f\|_q.$$

With this definition it is clear that strong boundedness on L^q implies boundedness on L^q, which implies weak boundedness on L^q, provided of course that the R_j's are uniformly bounded on L^q. Moreover, if T_* is strongly bounded on some L^{q_0} space and weakly bounded on L^{q_1}, then T_* is bounded on L^q for all q strictly between q_0 and q_1.

DEFINITION. A family T_{jv} of linear operators is essentially positive if there exist linear operators S_{jv} and U_{jv}, with $S_{jv} \geq 0$, $U_{jv} \geq 0$, S_* bounded on L^r, $1 < r \leq \infty$, and such that $T_{jv} = U_{jv} - S_{jv}$.

THEOREM 2. *Let $1 \leq p < 2$, and let T_{jv} be an essentially positive family of linear operators. Suppose there exists a $q \neq p$ (we shall assume $q > p$) such that T_* is strongly bounded on L^q, and suppose there exists an $\varepsilon > 0$ such that $\sup_j \| \sup_v |T_{jv} f| \|_r \leq C_r \|f\|_r$ and $\|(\sum |R_j f|^2)^{1/2}\|_r \leq C_r \|f\|_r$ for r in $(p, p+\varepsilon)$. Then T_* is bounded on L^r for all r in $(p, q]$.*

REMARK. There is a similar but simpler principle when $p \geq 2$, whose statement and proof we omit.

PROOF. We first assume that for all but finitely many j, $T_{jv} = 0$ for all v, and we shall obtain a bound for T_* independent of this finite number N. So, fixing an r with $p < r < q$, we may assume that $\|T_* f\|_r \leq A(N) \|f\|_r$.

We consider first inequalities of the form

$$(*) \qquad \left\| \left\| \sup_v |T_{jv} g_j| \right\|_{l^s} \right\|_{L^t} \leq C_{s,t} \| \, \|g_j\|_{l^s} \|_{L^t}.$$

By assumption, $(*)$ holds for $s = t$ in $(p, p+\varepsilon)$. It also holds with $t = r$ and $s = \infty$, with constant $B(N)$ depending on $A(N)$ since U_{jv} and S_{jv} are positive. Thus by interpolation there exists an $\tilde{r}$, $p < \tilde{r} < r$, such that $(*)$

holds for $t = \tilde{r}$ and $s = 2$. Now,

$$\left\| \sup_{jv} |T_{jv} R_{j+k} f| \right\|_{\tilde{r}} \leq \left\| \left(\sum_j \sup_v |T_{jv} R_{j+k} f|^2 \right)^{1/2} \right\|_{\tilde{r}}$$
$$\leq D(N) \left\| \left(\sum_j |R_{j+k} f|^2 \right)^{1/2} \right\|_{\tilde{r}}$$
$$\leq C_r D(N) \|f\|_{\tilde{r}}.$$

So T_* is weakly bounded on $L^{\tilde{r}}$ and hence by the comment preceding the theorem it is bounded on L^r with constant $E(N)$. However, keeping track of constants, we see that for some $0 < t < 1$ and some numbers a and b, $E(N) \leq a + bA(N)^t$. Thus $A(N) \leq c$, for some c independent of N, concluding the proof of the theorem.

3. An auxiliary proposition. We now specialise to operators of the form $(T_{jt} f)^\wedge(\xi) = m(2^j t\xi) f^\wedge(\xi)$ for $j \in Z$ and $1 \leq t \leq 2$. We wish to apply Theorem 2 in the case $q = 2$, and so we need simple criteria for determining when a maximal operator is bounded on L^2, and when a maximal operator of the form $\sup_{1 \leq t \leq 2} |K_t * f|$ is bounded on L^p.

PROPOSITION. *Let $K^\wedge = m \in L^\infty$. Then*
(i) $\| \sup_{0<t<\infty} |K_t * f| \|_2 \leq C \|f\|_2$ *if for some $\alpha > \frac{1}{2}$ we have*

$$\sup_{w \in S^{n-1}} \left(\int_0^\infty |u^{\alpha+1} (d/du)^\alpha [u^{-1} m(uw)]|^2 \, du/u \right)^{1/2} < \infty.$$

(ii) $\| \sup_{1 \leq t \leq 2} |K_t * f| \|_p \leq C \|f\|_p$ *if for some $\alpha > 1/p$ (or $\alpha = 1$ if $p = 1$) both m and $(\xi \cdot \nabla)^\alpha m$ are L^p multipliers.*

REMARKS. (i) Here, $(d/du)^\alpha$ is the fractional differentiation operator defined for example in [**3**], and

$$(\xi \cdot \nabla)^\alpha m(\xi) = (d/du)^\alpha m(u\xi)|_{u=1} = \int (2\pi i x \cdot \xi)^\alpha K(x) e^{2\pi i x \cdot \xi} \, dx.$$

(ii) When m is radial, part (i) of the proposition is in [**3**]. Other similar criteria for L^2 boundedness of maximal operators appear in [**1, 5** and **7**].

PROOF. Write

$$\frac{m(t\xi)}{t} = C_\alpha \int_0^\infty (u - t)_+^{\alpha-1} (d/du)^\alpha [m(u\xi)/u] \, du.$$

Therefore,

$$|K_t * f| \leq C_\alpha \int_0^\infty (1 - t/u)_+^{\alpha-1} t/u |P_u^\alpha f| \, du/u,$$

where $(P_u^\alpha f)^\wedge(\xi) = u^{\alpha+1} (d/du)^\alpha [m(u\xi)/u] \, f^\wedge(\xi)$. Thus, if $p = 2$ and $\alpha > \frac{1}{2}$,

$$\sup_{0<t<\infty} |K_t * f| \leq C_\alpha \left(\int_0^\infty |P_u^\alpha f|^2 \, du/u \right)^{1/2},$$

and so

$$\left\| \sup_{0<t<\infty} |K_t * f| \right\|_2 \le C_\alpha \|f\|_2$$

if the hypothesis of part (i) is fulfilled. If $p \neq 2$ and $t \geq 1$,

$$\begin{aligned} |K_t * f| &\le C_\alpha \left(\int_1^\infty |(1-t/u)_+^{\alpha-1} t/u|^q \, du \right)^{1/q} \left(\int_1^\infty |P_u^\alpha f|^p \, du/u^p \right)^{1/p} \\ &\le C_\alpha t^{1/q} \left(\int_1^\infty |P_u^\alpha f|^p \, du/u^p \right)^{1/p}, \quad \text{if } 1/p + 1/q = 1 \text{ and } \alpha > 1/p. \end{aligned}$$

Hence

$$\left\| \sup_{1\le t\le 2} |K_t * f| \right\|_p \le C_\alpha \left(\int_1^\infty \|P_u^\alpha f\|_p^p \, du/u^p \right)^{1/p}.$$

But the L^p operator norm of P^α is controlled by the L^p multiplier norms of m and $(\xi \cdot \nabla)^\alpha m$.

4. Proof of Theorem 1. At this point we shall assume that the reader is familiar with the contents of [**1**], where, amongst other things, Bourgain proves that there exist a number $L = L(B)$ and an $A \in \mathrm{SL}(n, R)$ such that if $K = \chi_{A(B)}$, then

(a) $|K^\wedge(\xi)| \le C(|\xi|L)^{-1}$,
(b) $|K^\wedge(\xi) - 1| \le C|\xi|L$,
(c) $|\xi \cdot \nabla K^\wedge(\xi)| \le C$,

with C an absolute constant.

We shall obtain a Littlewood-Paley decomposition of R^n, $I = \sum R_j$, such that $\|(\sum |R_j f|^2)^{1/2}\|_p \le C_p \|f\|_p$, $1 < p \le 2$, with C_p independent of n, then observe that the essentially positive family of operators $(K - P_L)_{t>0}$ has maximal function strongly bounded on L^2 with respect to this decomposition, with constant independent of everything. Here P is the Poisson kernel; it is of course true that P_* is bounded on $L^p(R^n)$, $1 < p \le \infty$ with constant independent of n (see [**8**]). Finally we show that $\| \sup_{1\le t\le 2} |K_t * f| \|_p \le C_p \|f\|_p$, with C_p independent of B and n if $p > 3/2$, which will conclude the proof once we apply Theorem 2; for the case of $\tilde{M}$ this third step is not required since $\|K\|_1 = 1$.

Each of these steps is easy; for the first we merely take $R_j = P_{2^{j+1}} - P_{2^j}$; then $(\sum |R_j f|^2)^{1/2} \le (\log 2)^{1/2} g_1(f)(x)$, g_1 being the classical Littlewood-Paley function, which Stein has shown [**9**] to satisfy $\|g_1(f)\|_p \le C_p \|f\|_p$, $1 < p \le 2$, with C_p independent of n.

For the second step, one may apply part (i) of the proposition to each of the operators $(K - P_L)R_k$, $k \in Z$, using (a)–(c) to obtain $\|[K - P_L]R_{k*}\|_2 \le C a_k \|f\|_2$, with $\sum a_k^t < \infty$, $0 < t \le 1$. This is not exactly what being strongly bounded on L^2 means, but a slight modification of this argument will give precisely what we require.

Finally, observe that $K^\wedge$ has L^1 multiplier norm 1, and by (c) above $(\xi \cdot \nabla)K^\wedge$ has L^2 multiplier norm dominated by an absolute constant; after setting up the appropriate complex-analytic interpolation argument, one obtains that $(\xi \cdot \nabla)^\alpha K^\wedge$ has L^p multiplier norm dominated by an absolute constant if $\alpha < 2/p'$, $0 < \alpha < 1$, $1 < p < 2$. An application of part (ii) of the proposition yields $\| \sup_{1 \leq t \leq 2} |K_t * f| \|_p \leq C_p \|f\|_p$, with C_p depending only on p if $1/p < 2/p'$, which is $p > 3/2$.

5. Concluding remark. The reader will observe that only the last of the three steps does not work for all $p > 1$; if the method is to succeed further, results of the form $(\xi \cdot \nabla) K^\wedge$ having L^p multipler norm not depending on B or n, $p \neq 2$, would be useful. Of course the L^1 multiplier norm of this operator is essentially n. Is it possible to do better than interpolation between $p = 1$ and $p = 2$ for this operator? Such results, if true, would give a new expression to the philosophy that, for large n, "most of the mass of a convex body is situated near its boundary".

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