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A RATIONAL BILLIARD FLOW IS UNIQUELY ERGODIC IN ALMOST EVERY DIRECTION

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The nature of the long-term behavior of a billiard ball moving on a frictionless table is a question with strong intuitive appeal. A billiard flow is a type of geodesic flow. Geodesic flows, and billiard flows in particular, have provided important examples in the field of dynamical systems.

Let Q be a planar polygon. One can define a geodesic flow f_t on the unit tangent bundle $U(Q)$ so that orbits of this flow project to billiard ball paths on Q. The polygon Q is said to be *rational* if all of the angles of Q are rational multiples of π. When Q is rational the tangent vectors to a given orbit are parallel to a finite set of unit vectors. The orbits with initial direction θ lie in an invariant surface M_θ. M_θ consists of a finite number of copies of Q, one for each potential direction of an orbit with initial direction θ (cf. [**F-K**]). The dynamical analysis of f_t breaks up into an analysis of the flows $f_t|M_\theta$ as θ varies.

The results of [**Z-K**] and [**B-K-M**] show that for a typical direction θ the flow is minimal, i.e. all orbits are dense. The purpose of this note is to announce the following

THEOREM 1. *For almost every θ the flow $f_t|M_\theta$ is uniquely ergodic.*

A flow is *ergodic* with respect to a probability measure if every invariant set has measure zero or one. A flow is *uniquely ergodic* if there is precisely one invariant probability measure. A uniquely ergodic flow is ergodic with respect to its unique invariant measure. The surfaces M_θ described above have natural invariant measures coming from Lebesgue measure on Q. Theorem 1 implies that the billiard flows are ergodic in almost every direction with respect to the natural invariant measures.

Unique ergodicity can also be described in terms of the distribution of orbits. An orbit is *uniformly distributed* with respect to a probablity measure μ if for every open set V with $\mu(\partial V) = 0$ the orbit visits V with an asymptotic frequency of $\mu(V)$. A flow is uniquely ergodic if every orbit is uniformly distributed. The natural measures on the sets M_θ project to Lebesgue measure on Q. These remarks imply the following

COROLLARY 1. *For almost every θ the projection to Q of every orbit with initial direction θ is uniformly distributed in Q.*

Theorem 1 has consequences for billiard tables which do not have rational angles. The set of all polygons with a given number of sides forms an open

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subset of a finite-dimensional vector space. We would like to thank A. Katok and M. Boshernitzan for pointing out the following corollary to Theorem 1.

COROLLARY 2. *There is a dense G_δ consisting of polygons for which the billiard flow, f_t, is ergodic.*

If Q is a rectangle or, more generally, if reflections through the sides of Q generate a tesselation of the plane then Theorem 1 is a consequence of Weyl's analysis of toral flows, as is pointed out in [**F-K**].

If the affine group generated by reflections in the sides of Q acts discretely on the plane then Theorem 1 follows from results in [**B** and **G**].

Theorem 1 follows from a result that we prove about Riemann surfaces and quadratic differentials. A quadratic differential q determines a "real foliation" defined by $\operatorname{Re} q^{1/2}\,dz = 0$. This foliation admits a transverse invariant measure. If it admits precisely one such measure we say that it is uniquely ergodic.

THEOREM 2. *Given a compact Riemann surface M and a holomorphic quadratic differential q then for almost all θ the real foliation of $e^{i\theta}q$ is uniquely ergodic.*

The unique ergodicity of almost all interval exchanges proved in [**M** and **V**] follows from Theorem 2. The proof of Theorem 2 uses a result from [**M**] relating the unique ergodicity of q to the asymptotic behavior of q under the Teichmüller flow.

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