should make a valuable, easily accessible reference work. Much of the material covered is due to the author and his students, and, except for some overlap with the recent book [2] of Karpilovsky, most of it cannot be found in other books.

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Differential systems involving impulses, by S. G. Pandit and S. G. Deo, Lecture Notes in Mathematics, Vol. 954, Springer-Verlag, Berlin, 1982, vi + 102 pp., \$8.00. ISBN 3-5401-1606-0

This book presents and studies a new class of generalized ordinary differential equations containing impulsive terms linearly. The review first introduces the discipline of generalized ODE, briefly describes the contents of the book, and offers comments on the treatment in the present literature. Readers familiar with the background, and those who do not believe that a book review is an excuse for an expository paper, may wish to begin around equation (11).

Consider ordinary differential equations in R^n,

$$\dot{x} = f(t, x) \quad (\text{i.e., } dx(t)/dt = f(t, x(t))). \tag{1}$$

If the right side, $f\colon R^{1+n} \to R^n$, is continuous, it is perhaps obvious what the solutions $x(\cdot)$ of (1) ought to be: an explicit definition is almost superfluous.

Applications soon dictated that continuity of f be relaxed. One studies 'block box systems'

$$\dot{x} = Ax + bu(t) \tag{2}$$

by examining the 'responses' $x(\cdot)$ to various 'inputs' $u(\cdot)$; and Laplace transform methods suggest that it is the discontinuous inputs that are crucial: e.g., a signum function, a unit stepfunction, or even a delta "function".

With discontinuities present in the term $u(\cdot)$ (the forcing term, or control), one can no longer successfully require that solutions $x(\cdot)$ satisfy the differential equation (2) for all t. Several plausible definitions of generalized solution come to mind:

(A) functions $x(\cdot)$ that are absolutely continuous (locally) and satisfy (2) for almost all t (absolute continuity cannot be relaxed to ordinary continuity without theory breaking down: even $\dot{x} = 0$ would have too many solutions with $x(0) = 0$);

(B) functions $x(\cdot)$ such that

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}bu(s)\,ds$$

(the variation-of-constants formula is elevated to a definition);

(C) continuous functions $x(\cdot)$ whose distributional derivative satisfies (2) (in another version Schwartz's distributions might be replaced by Mikusiński operators).

Perhaps it is not surprising that these definitions turn out to be equivalent, at least for (2) with A, B constant. In 1918 C. Carathéodory discovered a far-reaching extension: definition (A) applies not only to (2), but also to nonlinear equations (1) as long as $f(t, x)$ is measurable in t and continuous in x (with a technical growth requirement added—the Carathéodory conditions [**1**, **2**]). A conceptually minor, but practically important, further generalization involves differential inclusions (orientor fields, differential inequalities when state space dimension $n = 1$). These are symbolically indicated by

$$\dot{x} \in F(t, x), \tag{3}$$

where $F(t, x)$ is a subset of R^n varying with t, x and satisfying suitable conditions. For example, $\dot{x} \in [-1, 1]$ has, as classical solutions, differentiable real-valued functions x satisfying $|\dot{x}(t)| \leqslant 1$ for all t, and, as Carathéodory solutions, the real functions with Lipschitz constant 1. (These concepts, dating back to 1935, are due to A. Marchaud and S. K. Zaremba; for a survey and references see [**13**].)

For some time there was no compelling reason to go any further: most ODE which came from applications did have the right side depending continuously on the state variable x. In addition, there was an obvious obstacle to cheap generalisation—the initial value problem

$$\dot{x} = \begin{cases} -1 & \text{for } x \geqslant 0, \\ 1 & \text{for } x < 0, \end{cases} \qquad x(0) = 0,$$

has no solution on any interval $[0, \varepsilon)$, $\varepsilon > 0$: a kind of jamming occurs at the 'endpoint' $x = 0$.

During and after the second world war, new mathematical problems became more frequent and important in this area: for example, automatic control systems, discontinuous feedback control (home thermostats are an obvious instance), and optimal control. The impetus had been present at least since 1868 when Maxwell published a paper [**11**] titled *On governors* (the governor of a Watt steam engine being a prime example). These governors, and other systems involving 'automatic regulation or control', signal 'amplifiers' (such as the diode oscillator leading to the van der Pol equation), and power-assisted turrets in World War II bombers, were all devices involving continuous state feedback. In the case of linear or proportional feedback, the analysis was simplified (positive or negative feedback for dimension 1), but often nonlinearities were essential to the situation. Even more intractable mathematically were the far simpler and cheaper two-position on-off controllers. Studies of these, presumably with application to the guidance systems in the V1 and V2 weapons, led to the first attempt at systematic treatment [**6**].

To illustrate the genesis of equations discontinuous in the state, consider a particularly simple example: time-optimal control of a service trolley [**9**]. The kinematic equations, in R^2 as state space, are

$$\dot{x} = y, \quad \dot{y} = u, \qquad -1 \leqslant u(t) \leqslant 1. \tag{4}$$

The time-optimal control problem is the following: given initial values $(x_0, y_0) \in R^2$, find an 'optimal control' $u:[0, +\infty) \to [-1, 1]$ such that the solution $x(\cdot)$, $y(\cdot)$ of (4), with these initial data and the chosen forcing term $u(\cdot)$, reaches the origin (i.e., $x(T) = 0 = y(T)$) at minimal time $T > 0$. Heuristics (involving controls with constant values ± 1) and elementary methods suggest that there is a significant 'switch curve' in R^2, $y = (-\operatorname{sgn} x) \cdot \sqrt{2|x|}$, and the optimal control u should have a value of -1 above this curve (and on its left branch $y > 0$), a value of 1 below the curve (and on its right branch $y < 0$), and a value of 0 at the origin (loc. cit., p. 10).

Now, if this prescription is used to define u: $R^2 \to [-1, 1]$ as a function of the state, $u = u(x, y)$, there arises quite naturally the feedback system corresponding to (4),

$$\dot{x} = y, \qquad \dot{y} = u(x, y); \tag{5}$$

obviously, this is a differential system, autonomous but nonlinear, with right sides depending discontinuously on the state variables. Having constructed this 'synthesis of the optimal regime', one is confronted with the obvious questions: Does (5) have existence of solutions to the initial value problem? (Probably yes, by construction: the time-optimal solutions should satisfy (5).) Does (5) have uniqueness? (In other words, will a mechanism which implements the constructed on-off feedback $u(x, y)$ actualy perform as expected?) In the positive case does (5) even exhibit continuous dependence on initial data (insensitivity to erros in measurement)? Neither the classical nor Carathéodory theories provide answers. Since $u(x, y)$ is discontinuous, there is no question of a Lipschitz condition, and little use for any of the more refined uniqueness tests.

This difficulty—lack of an adequate theory for discontinuous ODE—became more acute with the appearance of differential games. Here one studies systems with kinematic equations $\dot{x} = f(x, u, v)$, wherein the controls $u(\cdot)$, $v(\cdot)$ are to be suitably chosen by two 'players' who may have inconsistent or even opposed goals [**12, 8**]. Even if f is reasonably smooth, the players might well choose control functions discontinuous in the state variable x, led by optimality considerations analogous to those in (4).

Probably the first systematic attempt to treat a wide class of possibly discontinuous equations was made by Filippov [**5**]; others then followed. For the sake of illustration, let us describe one of these concepts, due to Krasovskij (a more detailed account appears in [**7**]). Assume equation (1) is given; associate with it the differential inclusion (3), where F is obtained from f as follows:

$$F(t, x) = \bigcap_{k=1}^{\infty} \overline{\operatorname{cvx}} f(t, x + k^{-1}B). \tag{6}$$

Here B is the unit ball in R^n (so that $x + k^{-1}B$ is the ball of radius k^{-1} about x), and $\overline{\text{cvx}}M$ denotes the closure of the convex hull of M. Then 'solutions' of (3), i.e., locally absolutely continuous functions $u(\cdot)$ which satisfy the differential inclusion (3) almost everywhere, are defined to be the Krasovskij generalised solutions of (1). Two properties are immediate: if $f(t, x)$ is continuous in x, then each $F(t, x) = \{f(t, x)\}$ is a singleton, and the present definition reduces to a previous one; second, each Carathéodory solution of (6) is a Krasovskij solution. (The so-called Filippov solutions have only the former property.)

A moderately successful theory of generalised solutions is now available (e.g., a positive answer to the converse problem in the synthesis of time-optimal feedback for controllable linear systems (9) with one-dimensional controls [7, Theorem 9.4]). Some obvious questions have still not been addressed. One involves integration theory: differential equationists will always view this as the study of (generalised) solutions of the "trivial" equation (1) with x absent, or, better, of this equation 'made autonomous' by the familiar device of raising state dimension by 1:

$$\dot{x} = f(\theta), \qquad \dot{\theta} = 1. \tag{7}$$

There appear to be close relations between the Krasovskij solutions of system (7) and the concept of belated integrals in the sense of McShane (or, better, Itô-belated [**10**]). A second question concerns the 'other' generalisation of solutions to differential equations: the weak solutions of linear partial differential equations.

The reader may have noticed by now that the prime problem is not really how the solution concept should be generalised, but rather, what is the class of differential equations to be treated by these solutions. What is more important is that the impetus for studying broader classes of equations should come from applications (rather than from pure ODE theory).

To come closer to the topic of the reviewed book, in control-theoretic texts often the most natural and immediate interpretation of the physical laws governing the process is a differential equation of the form

$$x^{(n)} + a_1x^{(n-1)} + \cdots + a_nx = b_0u^{(m)} + b_1u^{(m-1)} + \cdots + b_mu \tag{8}$$

for an unknown function $x(\cdot)$, and involving a function $u(\cdot)$ and its derivatives (with $u(\cdot)$ given or arbitrarily taken from a suitable class of functions; numerator dynamics; realisation of transfer functions). If $n > m$ and the a_i, b_j are constant, this may be rewritten in vector form as

$$\dot{x} = Ax + Bu, \tag{9}$$

where the new vector function $x(\cdot)$ has the old scalar unknown $x(\cdot)$ as first coordinate; similarly, for $u(\cdot)$. (Actually $n > m$ may be relaxed to allow $n = m$. Indeed, a vector equation

$$\dot{x} = Ax + Bu + C\dot{u} \tag{10}$$

may be written, using $y := x - Cu$, as $\dot{y} = Ay + (B + AC - \dot{C})u$ in the form of (9); e.g., the scalar equation $\ddot{x} = \ddot{u} + u$ obviously reduces to $\ddot{y} = u$ with $y := x - u$, and one may recover the observation x from the new states y.)

In (10) we have implicitly assumed that the data A, B, C are constant or depend only on t. The reviewed book proposes to treat a nonlinear analogue of this, the most general being

$$\dot{x} = f(t, x) + g(t, x)\dot{u}, \tag{11}$$

$$\dot{x} = f(t, x, u) + g(t, u)\dot{u}. \tag{12}$$

See (2.1) and (2.32) (unqualified references are to the reviewed book). Here the functions $f(\cdot)$, $g(\cdot)$ are to satisfy reasonable assumptions; $u = u(t)$ is a right-continuous vector function of bounded variation; and solutions $x(\cdot)$ of (11) are defined to be right-continuous functions of bounded variation which satisfy (11) when $\dot{x}$, $\dot{u}$ denote derivatives in the sense of Schwartz distributions (locally on compact intervals, Definition 1.4). Thus (11) indicates a new class of ordinary differential equations, and the solution concept proceeds in the direction (C) suggested at the beginning of this review. Intuitively, in (11) the function $u(\cdot)$ is allowed to have discontinuities, so $\dot{u}$ has impulsive behavior, and, therefore, $x(\cdot)$ may be discontinuous.

In the book under review, Chapter 1 introduces needed concepts (BV, complex measures, distributions) and treats one example—a growth problem. Chapter 2 is concerned with existence and uniqueness of the so-defined solutions for the initial value problem of (11) with $u(\cdot)$ given, including an 'integral representation'

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))\, ds + \int_{t_0}^{t} g(s, x(s))\, du(s), \tag{13}$$

and also existence of cost-optimal solutions for the corresponding control problem (12). Chapter 3 ("Stability and asymptotic equivalence") interprets the quasi-linear problem $\dot{x} = A(t)x + F(t, x)\dot{u}$ as a perturbation of the linear ODE $\dot{x} = A(t)x$. Chapter 4 ("Impulsive systems") proceeds analogously, except that the unperturbed system is taken to be bilinear: $\dot{x} = A(t)x\dot{u}$ (A n-square, u scalar). The final chapter ("Lyapunov's Second Method") extends stability concepts and methods to $\dot{x} = f(t, x) + g(t, x)\dot{u}$, again viewed as a perturbation of a basic ODE $\dot{x} = f(t, x)$. Each chapter concludes with historical notes. The Bibliography contains 52 references; of these some 33 are devoted to the topic of impulses or measure terms in ODE, 7 are general references, 8 are due to one or both the authors, 6 are unpublished, and 11 are not referred to in the text. According to the Preface, the book is "...an attempt to unify the results of several research papers published during the last fifteen years."

The reviewer's comments follow.

We mention minor matters first. The bibliography is careless. The names Halanay, Tesei, and Zavališčin are consistently written as Halany, Texi, and Zabalischin (the last is probably a further misspelling of a misspelling, Zabalishchin, in reference **[22]** of the book). Jürgen Groh is listed under "J" as Groh Jirgen, but otherwise not referred to. Two items are ominous, but may have local effect only. The first involves the statement and proof of Theorem 4.2; the assertion, with de-encumbered notation, is that, if in $Mx = a$ the matrix M is singular and $a \neq 0$, then no solution x exists (see the sentence beginning on the last line of p. 59). In Theorem 2.5 on the existence of optimal controls and solutions, any assumption that would ensure that the limit of "suboptimal" controls is necessarily an admissible control has been omitted; as stated, the theorem is false trivially.

The subject, study of equations (11), is completely lacking in motivation. This is particularly damning in a discipline like differential equation theory, which is almost exclusively devoted to, and justified by, the solution of problems which come from the outside. (ODE theory has little inner harmony and beauty; it is a far more obvious example than group theory of a discipline that will not appear in a Bourbaki treatment.) Of course, there cannot be any good linear examples: As shown, (10) can always be transformed to the classical case of equations without 'impulses'.

In fact, some applications are mentioned in the Preface. One involves the 'control problem' $\dot{x} = f(t, x, u)$: "Suppose that the control function $u(t)$...[is] of bounded variation... then the solution $x(t)$... may possess discontinuities." This is false (or meaningless). The "Case and Blaquière problem" and Itô's equation have been pasted in with no attempt to relate to, or actually treat by, the methods of the book.

Finally, we come to a disaster—the "growth problem" (Preface and pp. 8–9). The simplified initial value problem is

$$\dot{x} = x \cdot \dot{u}, \qquad x(0) = 1, \tag{14}$$

with

$$u(t) = t + aH(t - t_1), \qquad H(t) = \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t \geqslant 0. \end{cases}$$

The interpretation is growth of a single species $x(t)$, governed by $\dot{x} = \alpha x$ with reproduction rate $\alpha > 0$ (instantaneous growth rate per unit time): "After some...time intervals t_1...fish are removed...and new...fish...released.... The growth of the fish population is impulsive." This is modelled by using an impulsive control $\alpha = \dot{u}$ of the reproduction rate α! Of course, this is possible, but it grates against the sensibilities of any applied mathematician.

Unfortunately, worse is to come (other than the misprints $\dot{u} = 1 + aH(t_1)$ on p. 9, line 19). One solves (14) by observing that $\dot{x}/x = \dot{u}$, so $x(t) = e^{u(t)} \cdot 1$,

$$x(t) = \begin{cases} e^t & \text{for } 0 \leqslant t < t_1, \\ e^t \cdot e^a & \text{for } t_1 \leqslant t. \end{cases} \tag{15}$$

If one insists on greater sophistication (better, robustness), one approximates the delta function and takes limits of approximate solutions; this confirms the exhibited result. The book provides the following "solution" (p. 9, (4)):

$$x(t) = e^t/(1 - a) \quad \text{for } t_1 \leqslant t, \tag{16}$$

and if $a = 1$, the impulsive equation (13) has no "solution". That this is put forward intentionally is confirmed by the treatment of the more general vector equation

$$\dot{x} = A(t)x\dot{u}, \qquad u(t) = t + \textstyle\sum a_k H_k(t)$$

in Chapter 4, Theorem 2.4.

What led the authors to a theory which has the consequence $e^a = 1/(1 - a)$ (for the coefficient of e^t above)? How did this happen? What was the initial error? The author of this review had thought that mathematics is self-correcting, that errors of fact—even if not caught by the author, referees, reviewer, or fellow workers—tend to be instinctively ignored by the mathematical community, and that questionable foundations are never built upon to create a tottering construction. This book provides a counterexample; and the story goes back to 1971 and [3].

First, then, the error. The authors believe that every distribution (say, on R^1) is a complex measure—i.e., a continuous linear functional on the space C_c^∞ (infinitely differentiable functions $R^1 \to R^1$ with compact support) endowed with the topology defined by the supremum norm—and, simultaneously, that every distribution has a derivative, in the sense of Schwartz distributions. (See p. 6, lines 9–10 and p. 7, lines 2–6; also see [**3**, p. 44] and [**4**, p. 155].)

Second, the reason. In (11), $\dot{x}$ and $\dot{u}$ are to be distributional derivatives (of right-continuous BV functions), i.e., distributions. Also in (11) one has the term $g(t, x(t), u(t)) \cdot \dot{u}$, i.e., a product of a discontinuous function with a distribution. Thus one needs objects which can be both differentiated and multiplied. Schwartz distributions do not function this way. One can multiply locally integrable functions with complex measures, but not with derivatives. The authors' solution concept is not a valid definition.

Third, the details. Suppose that for (14), with $u = t + aH$, one guesses the form $x = e^t(1 + cH)$ with unknown constant c; substitution leads to

$$c\delta = a\delta + acH\delta.$$

If one now pretends that the product $H\delta$ is δ (which it is not), there results $c = a/(1 - a)$ and (16).

The discrepancy may seem minor: distributions are indeed continuous linear functionals on C_c^∞, but under a different topology from that of the sup norm. It is true that many differential equationists view topology with disdain (for this, one must ignore the work of Poincaré, Lefschetz, and Pontrjagin): here it leads to the consequence $e^a = 1/(1 - a)$.

Suppose that the concept had instead been based on the integral formula (13), and the book written without any mention of impulses, distributions, and integral representation (necessarily, with a changed title). Indeed, some of the references take this point of view. Suppose also that the reviewer were permitted to comment on what then might have been, rather than properly confining himself to the book as written. What remains of the main objections is the $e^a = 1/(1 - a)$ paradox and the lack of motivation; these are connected.

To reemphasize, the problem (14) has the robust solution (15) and no others; the corresponding integral version

$$x(t) = 1 + \int_0^t x(s)\,du(s), \qquad u(t) = t + aH(t - t_1),$$

has the solution (16) and no others. The discrepancy extends, of course, to the general situation (11) and the corresponding Itô-type version $dx = f(t, x)\,dt + g(t, x)\,du$.

Why should one study (11) at all? If this is answered satisfactorily (and the reviewer believes it might), why should one adopt (13) as a definition of solution (especially because it leads to discrepancies)?

This book, and part of the literature on impulsive ODE, are fundamentally flawed.

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Einhüllende Algebren halbeinfacher Lie-Algebren, by Jens C. Jantzen, Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge · Band 3, A Series of Modern Surveys in Mathematics, Springer-Verlag, Berlin, 1983, 298 pp., DM 118; Approx. U.S. $45.80. ISBN 3-5401-2178-1

One of the fundamental problems in abstract harmonic analysis is the determination of the set of (equivalence classes of) irreducible unitary representations of a topological group G. These are continuous homomorphisms of G into the group of unitary operators on a Hilbert space; one assumes, in addition, that the Hilbert space has no nontrivial closed subspaces invariant under the whole group. This is a nonlinear problem, in the sense that group elements and unitary operators can be multiplied, but not added. It is tempting to look for ways to linearize things, for example because of the great success that idea enjoys in the elementary representation theory of finite groups. (There one considers the convolution algebra of all functions on the group. The