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LOCAL MODULI FOR MEROMORPHIC DIFFERENTIAL EQUATIONS

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1. Introduction. This note announces results concerning the parametrization, in the sense of (local) moduli, of the equivalence classes of systems of meromorphic differential equations of the form

$$(*) \qquad du/dz = Au$$

near an irregular singular point (assumed to be $z = 0$). Here u is an n-component column vector, A is an $n \times n$ matrix of meromorphic functions, and equivalence of systems defined by matrices A and B means that there is a meromorphic invertible $n \times n$ matrix x such that

$$(**) \qquad x[A] \stackrel{\text{def}}{=} xAx^{-1} + (dx/dz)x^{-1} = B$$

near $z = 0$. If $\mathcal{F}_{\text{cgt}}$ (resp. $\mathcal{F}$) is the field of quotients of the ring of convergent (resp. formal) power series in z with coefficients in $\mathbf{C}$, $(**)$ defines an action of $\mathrm{GL}(n, \mathcal{F}_{\text{cgt}})$ on $\mathfrak{gl}(n, \mathcal{F}_{\text{cgt}})$, reflecting the fact that $(*)$ goes over to the system $dv/dz = Bv$ under the substitution $v = xu$; replacing $\mathcal{F}_{\text{cgt}}$ by $\mathcal{F}$ leads to the notion of formal equivalence. We note that for any commutative ring R (with unit) equipped with a derivation D, $(**)$ defines an action of $\mathrm{GL}(n, R)$ on $\mathfrak{gl}(n, R)$, with D replacing d/dz; if R is a suitably restricted ring of Laurent series in z with coefficients in the ring of convergent power series in d variables and $D = d/dz$, we obtain the notion of equivalence of analytic families of systems $(*)$ depending on d parameters, which is basic to the theory of local moduli (cf. [**BV2**]).

One parametrizes the equivalence classes of systems $(*)$ in two steps. The first step is the classification up to formal equivalence, i.e., the description of the orbit space $\mathrm{GL}(n, \mathcal{F})\backslash\mathfrak{gl}(n, \mathcal{F})$; the second step is to fix a formal class Ω with $\Omega_{\text{cgt}} \stackrel{\text{def}}{=} \Omega \cap \mathfrak{gl}(n, \mathcal{F}_{\text{cgt}}) \neq \emptyset$, and to classify the systems $(*)$ in Ω_{cgt} up to equivalence, i.e., to describe the orbit space $\mathrm{GL}(n, \mathcal{F}_{\text{cgt}})\backslash\Omega_{\text{cgt}}$. The description of $\mathrm{GL}(n, \mathcal{F})\backslash\mathfrak{gl}(n, \mathcal{F})$; goes back to Hukuhara and Turrittin (see [**BV1**] for extensive references) and is based on the notion of a canonical form. The classical method of studying the second question is based on the technique of Stokes lines and Stokes multipliers [**Bi**, **J**]. Recently this has been examined from a more modern, and essentially cohomological, point of view, notably by Malgrange [**Ma1**, **Ma2**], Sibuya [**S**], and Deligne (cf. [**Be**]). The present note continues this theme by studying the equivalence of analytic families of systems $(*)$ and is based in a fundamental way on the theory of formal equivalence over general rings developed in [**BV2**].

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For our purposes we define a *canonical form* to be an element of $\mathfrak{gl}(n, \mathcal{F}_{\text{cgt}})$ of the type

$$B = D_{r_1} z^{r_1} + \cdots + D_{r_m} z^{r_m} + z^{-1} C,$$

where (a) $r_1 < r_2 < \cdots < r_m < -1$, the r_i being integers, (b) $C, D_{r_1}, \ldots, D_{r_m}$ are elements of $\mathfrak{gl}(n, \mathbf{C})$ that commute with each other, (c) the D_{r_i} are nonzero and semisimple, and (d) the real parts of all the eigenvalues of C are in $[0, 1)$ (if $m = 0$, $B = z^{-1}C$). For Ω we take the $\mathrm{GL}(n, \mathcal{F})$-orbit of B. We put $\Omega(B) = \Omega_{\text{cgt}}$ and write $X(B)$ for the space $\mathrm{GL}(n, \mathcal{F}_{\text{cgt}}) \backslash \Omega(B)$. Our main results show (cf. §3) that $X(B)$ may be viewed in a natural way as a space of the form $G_B \backslash H(B)$, where $H(B)$ is an algebraic variety isomorphic to an affine space $\mathbf{C}^d$ and G_B is an algebraic subgroup of $\mathrm{GL}(n, \mathbf{C})$ acting morphically on $H(B)$, and that "local moduli" exist at the "good" points of this quotient space: the restriction to "good" points is essential even in the simplest cases. Our results may thus be viewed as a description of the analytic deformations of the meromorphic differential equations $du/dz = Au$ when one fixes all the formal invariants of the equation, at least when the point of $H(B)$ defined by A is "smooth and stable".

2. The Stokes sheaf St_B and the identification $\mathrm{GL}(n, \mathcal{F}_{\text{cgt}}) \backslash \Omega(B) \approx G_B \backslash H^1(\mathrm{St}_B)$. Fix B as in §1 and let $\Psi = \exp\{\sum_{1 \le j \le m} (r_j + 1)^{-1} D_{r_j} z^{r_j + 1}\}$. The *Stokes sheaf* St_B is the sheaf of (in general noncommutative) groups defined on the unit circle T as follows: for any open subset U of T, $\mathrm{St}_B(U)$ is the group of holomorphic maps of the sector $\Gamma(U) = \{z \in \mathbf{C}^\times \mid z|z|^{-1} \in U\}$ into $\mathrm{GL}(n, \mathbf{C})$ such that

$$\Psi g \Psi^{-1} \sim 1 \ (\Gamma(U)),$$
$$dg/dz = z^{-1}[C, g] \quad \text{on } \Gamma(U).$$

Here, the notation ~ 1 $(\Gamma(U))$ in (a) means that, for any closed arc $U' \subset U$ and any $r \geq 1$, we have $\Psi g \Psi^{-1} - 1 = O(|z|^r)$ as $z \to 0$ in $\Gamma(U')$, the O being uniform in $\Gamma(U')$. If U is an arc $\neq T$ and $z_U^C = \exp(\log_U z \cdot C)$, where $\log_U$ is a branch of the logarithm on $\Gamma(U)$, the map $g \to z_U^{-C} g z_U^C$ takes $\mathrm{St}_B(U)$ onto a unipotent algebraic subgroup of $\mathrm{GL}(n, \mathbf{C})$ which is independent of the choice of the logarithm. So all the $\mathrm{St}_B(U)$ become unipotent algebraic groups in a natural way. Consequently, if $\mathfrak{U} = (U_i)$ is a finite open covering of T by arcs $\neq T$, the set $C(\mathfrak{U} \colon \mathrm{St}_B) = \prod_i \mathrm{St}_B(U_i)$ becomes a unipotent algebraic group, the set $Z^1(\mathfrak{U} \colon \mathrm{St}_B)$ of Čech 1-cocycles becomes an affine variety on which $C(\mathfrak{U} \colon \mathrm{St}_B)$ acts, and the space of orbits can be naturally identified with $H^1(\mathfrak{U} \colon \mathrm{St}_B)$. As usual, $H^1(\mathrm{St}_B)$ is the union of all the $H^1(\mathfrak{U} \colon \mathrm{St}_B)$ as $\mathfrak{U}$ varies over the coverings as above. If G_B is the centralizer of $C, D_{r_1}, D_{r_2}, \ldots, D_{r_m}$ in $\mathrm{GL}(n, \mathbf{C})$, G_B acts on each $\mathrm{St}_B(U)$ by g, $u \to g[u] = gug^{-1}$, and hence on $H^1(\mathrm{St}_B)$. Our starting point is the following variant of a theorem of Sibuya-Malgrange ([**S, Mal**]; cf. also [**Maj**]).

PROPOSITION 1. *There is a natural map θ from $\Omega(B)$ to $G_B \backslash H^1(\mathrm{St}_B)$ that is constant on the orbits of* $\mathrm{GL}(n, \mathcal{F}_{\text{cgt}})$ *in $\Omega(B)$ and induces a bijection of $X(B)$ with $G_B \backslash H^1(\mathrm{St}_B)$.*

3. The main theorems. By an analytic family a in $\mathcal{F}_{\text{cgt}}$ we mean a family $\{a(\lambda)\}$ $(\lambda \in \Delta^q)$, where Δ^q is a polydisc in $\mathbf{C}^q$ centered at the origin,

$a(\lambda) \in \mathcal{F}_{\mathrm{cgt}}$ for all $\lambda \in \Delta^q$, and there is an integer $r \geq 1$ such that, for some holomorphic function a' on $\Delta^q \times \{z| \, |z| < \varepsilon\}$, $a(\lambda)$ is the element of $\mathcal{F}_{\mathrm{cgt}}$ defined by $z^{-r}a'(\lambda: z)$. This leads in an obvious way to the notion of analytic families in $\mathfrak{gl}(n, \mathcal{F}_{\mathrm{cgt}})$ and in $\mathrm{GL}(n, \mathcal{F}_{\mathrm{cgt}})$. If A and A_1 are analytic families in $\mathfrak{gl}(n, \mathcal{F}_{\mathrm{cgt}})$ defined over Δ^q, they are called *equivalent* if there is an analytic family x in $\mathrm{GL}(n, \mathcal{F}_{\mathrm{cgt}})$ such that $x(\lambda)[A(\lambda)] = A_1(\lambda)$ for all λ in some neighbourhood of the origin. An analytic family A in $\mathfrak{gl}(n, \mathcal{F}_{\mathrm{cgt}})$ is said to be *in* $\Omega(B)$ if $A(\lambda)$ is in $\Omega(B)$ for all λ in some neighbourhood of the origin.

Let Σ be the set of Laurent polynomials $\sigma = \sum_{1 \leq j \leq m} a_j z^{r_j}$, where a_j is any eigenvalue of D_{r_j}, $1 \leq j \leq m$. For $\sigma, \tau \in \Sigma$ with $\sigma \neq \tau$, let $q = q(\sigma, \tau) \leq -2$ be the order of $\sigma - \tau$, c_q the coefficient of z^q in $\sigma - \tau$, and let $S(\sigma, \tau)$ be the (finite) set of rays in $\mathbf{C}^\times$ where $\mathrm{Re}(c_q z^q)$ vanishes. The rays belonging to $\bigcup_{\sigma, \tau \in \Sigma, \sigma \neq \tau} S(\sigma, \tau)$ are called the *Stokes lines* of B. Let $\mathfrak{T}(B)$ denote the collection of all finite coverings $\mathfrak{U} = (U_i)$ of T by open arcs of length $\leq \pi/(|r_1| - 1)$ with the restriction that the ends of the arcs of length equal to $\pi/(|r_1| - 1)$ are not on any Stokes line.

THEOREM 1. (i) *$H^1(\mathrm{St}_B)$ can be given the structure of an algebraic variety which is natural in the following sense: for any $\mathfrak{U} \in \mathfrak{T}(B)$, $C(\mathfrak{U}: \mathrm{St}_B)$ acts freely on $Z^1(\mathfrak{U}: \mathrm{St}_B)$, $H^1(\mathfrak{U}: \mathrm{St}_B) = H^1(\mathrm{St}_B)$, and $H^1(\mathrm{St}_B)$ is the geometric quotient of $Z^1(\mathfrak{U}: \mathrm{St}_B)$ for this action (see* [**MF**] *for the notion of geometric quotient); moreover, there is a global cross section for this action.*

(ii) *$H^1(\mathrm{St}_B)$ is isomorphic to the affine space $\mathbf{C}^d$, where d is the irregularity of B in the sense of Malgrange (cf.* [**Be**, *pp.* 233, 238]).

(iii) *The action of G_B on $H^1(\mathrm{St}_B)$ is algebraic.*

A point $\gamma \in H^1(\mathrm{St}_B)$ is called *G_B-smooth* if there exists a G_B-invariant open set U containing γ such that the geometric quotient $G_B \backslash U$ exists in the category of complex analytic manifolds. Let $H^1(\mathrm{St}_B)^{\mathrm{sm}}$ be the G_B-invariant open set of G_B-smooth points. Let $Y = G_B \backslash H^1(\mathrm{St}_B), \pi$ the natural map $H^1(\mathrm{St}_B) \to Y$, and $Y^{\mathrm{sm}} = \pi(H^1(\mathrm{St}_B)^{\mathrm{sm}})$; Y is given the quotient topology. The sheaf of G_B-invariant analytic functions on $H^1(\mathrm{St}_B)$ defines a sheaf on Y and converts Y into a ringed space; and Y^{sm} is the open subset of points around which this ringed space looks like a complex manifold of dimension $r = d - \delta$, where δ is the maximum dimension of the G_B-orbits in $H^1(\mathrm{St}_B)$.

THEOREM 2. *Fix $\gamma \in H^1(\mathrm{St}_B)^{\mathrm{sm}}$. Let A be an analytic family of elements in $\Omega(B)$ defined over Δ^q such that $\theta(A(0)) = \pi(\gamma)$. Then $\mu(A)\colon \lambda \to \theta(A(\lambda))$ is an analytic map of a neighbourhood of the origin into a neighbourhood of $\pi(\gamma)$. If A_1 is another analytic family in $\Omega(B)$ defined over Δ^q such that $\mu(A) = \mu(A_1)$ in a neighbourhood of the origin, then A and A_1 are equivalent.*

The proof of this theorem relies heavily on one of the main results of [**BV2**].

THEOREM 3. *Let r be as defined earlier. Then we can find an analytic family in $\Omega(B)$ defined over Δ^r such that $\mu(A)$ is an analytic isomorphism of a neighbourhood of the origin in Δ^r with a neighbourhood of the point $\pi(\gamma)$. Any such family is universal in the following sense. If A_1 is any analytic family in $\Omega(B)$ defined over Δ^q with $\theta(A_1(0)) = \pi(\gamma)$, we can find an analytic map*

$\alpha: \Delta'^q \to \Delta'^r$ (primes denote concentric polydiscs) vanishing at the origin such that the families A_1 and $A \circ \alpha$ are equivalent.

If C is semisimple, G_B is reductive, so we are in the paradigm of Mumford [**MF**]. Let us call a point $\gamma \in H^1(\mathrm{St}_B)$ *stable* if its G_B-orbit is closed and has dimension δ, and let $H^1(\mathrm{St}_B)^{\mathrm{s}}$ be the set of stable points; it is G_B-invariant and Zariski open. The statement that $H^1(\mathrm{St}_B)^{\mathrm{s}} \neq \emptyset$ is equivalent to saying that the action of G_B on $H^1(\mathrm{St}_B)$ is generically stable (cf. [**MF**, p. 154]).

THEOREM 4. *Suppose C is semisimple and $H^1(\mathrm{St}_B)^{\mathrm{s}} \neq \emptyset$. Then $Y^{\mathrm{s}} = G_B \backslash H^1(\mathrm{St}_B)^{\mathrm{s}}$ is an irreducible quasi-affine variety of dimension r. If Γ is the set of points γ in $H^1(\mathrm{St}_B)^{\mathrm{s}}$ such that $\pi(\gamma)$ is a simple point in Y^{s}, then $\Gamma \subset H^1(\mathrm{St}_B)^{\mathrm{sm}}$, Γ is dense in $H^1(\mathrm{St}_B)$, and $G_B \backslash \Gamma$ is a complex manifold of dimension r.*

Already in simple examples such as the Bessel and Whittaker equations, nonsmooth and smooth nonstable points exist. In general, $G_B \backslash H^1(\mathrm{St}_B)^{\mathrm{sm}}$ will not be separated. When B is such that the restriction of C to each spectral subspace of $(D_{r_1}, \ldots, D_{r_m})$ has a simple spectrum, then stable points exist, $PG_B = G_B/\mathbf{C}^\times$ acts generically freely on $H^1(\mathrm{St}_B)$, and $r = d - n + 1$.

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