BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 6, Number 2, March 1982

A NOTE ON A PROBLEM OF SAFF AND VARGA CONCERNING THE DEGREE OF COMPLEX RATIONAL APPROXIMATION TO REAL VALUED FUNCTIONS

BY S. W. ELLACOTT[1]

Saff and Varga **[1977, 1978]** discovered the surprising fact that given a real valued function on $[-1, 1]$ one can sometimes obtain a better rational approximation to f of a given type by allowing complex coefficients than by restricting the coefficients to be real. In this note we point out a connection between this result and the "Trefethen effect" (Trefethen **[1981a, 1981b]**) of near circular error curves for best approximation on the unit disc, which as Trefethen has shown is in turn closely related to the Carathéodory-Fejér Theorem and its generalisation to meromorphic approximation due to Takagi.

We consider a compact simply connected set Ω in the complex plane which is symmetric about the real line (i.e. $z \in \Omega$ iff $\bar{z} \in \Omega$). We assume further that the complement of Ω is simply connected in the extended plane and that Ω is a Faber domain (see e.g. Gaier **[1980]**). Finally if ψ denotes the conformal mapping of the complement of the unit disc D onto the complement of Ω such that $\psi(\infty) = \infty$, $\lim_{w\to\infty} \psi(w)/w$ finite real and positive, we assume for simplicity that ψ can be extended continuously to the boundary of the disc. All these properties are, of course, satisfied for the main case of interest here, $\Omega = I = [-1, 1]$.

$\mathfrak{A}(\Omega)$ will denote the set of functions continuous on Ω and analytic at interior points, and $\mathfrak{A}^R(\Omega)$ the subspace of functions satisfying $f(\bar{z}) = \overline{f(z)}$. $\mathfrak{T}: \mathfrak{A}(D) \longrightarrow \mathfrak{A}(\Omega)$ is the Faber transform (Gaier **[1980]**); $E^C_{mn}(f; \Omega)$ the error of best (Chebyshev) approximation to $f \in \mathfrak{A}^R(\Omega)$ from $\mathfrak{R}_{mn}(\Omega)$, the set of type (m, n) rational functions with no poles in Ω, and $E^R_{mn}(f; \Omega)$ the corresponding error of best approximation from the subset of rational approximations with real coefficients.

Saff and Varga **[1977, 1978]** observed that it is possible to find $f \in \mathfrak{A}^R(I)$ for which $E^C_{nn}(f; I) < E^R_{nn}(f; I)$ (e.g. $f(x) = x^2$, $n = 1$). Some authors attacked the case $n = 1$ by largely geometric arguments and showed among other things

Received by the editors July 14, 1981.

1980 *Mathematics Subject Classification*. Primary 30E10; Secondary 41A20, 41A25, 41A50.

[1]Author partially supported by a Royal Society European Visiting Fellowship.

that for certain classes of functions $E^C_{11}(f; I)/E^R_{11}(f; I) \geqslant \frac{1}{2}$ (Ruttan [1977]; Bennet, Rudnick and Vaaler, [1979]). We consider more generally $m \geqslant n-1$ and obtain a lower bound for $E^C_{mn}(f; \Omega)/E^R_{mn}(f; \Omega)$ which, if $m \geqslant n$ and Ω is convex, is for many functions very close to $\frac{1}{2}$.

We introduce the space $\widetilde{\mathfrak{R}}_{mn}(D)$ which may be described as those functions of the form

$$r(z) = \frac{\sum_{j=-\infty}^{m} a_j z^j}{\sum_{j=0}^{n} b_j z^j}$$

where the denominator has no zeros in D and the numerator represents a function bounded on $S = \{z|\ |z| = 1\}$ and analytic on the complement of D. Define $\widetilde{E}^R_{mn}(f; S)$ and $\widetilde{E}^C_{mn}(f, S)$ for $f \in \mathfrak{A}^R(D)$ in the obvious way.

We now list some simple relationships which are either obvious or straightforward consequences of results given in Trefethen [1981b], Gutknecht [1981] or Ellacott [1981]. Here $g \in \mathfrak{A}^R(D)$.

(i) $\widetilde{E}^R_{mn}(g; S) = \widetilde{E}^C_{mn}(g; S)$. From now on we drop the superscripts R and C when referring to this error,

(ii) $\mathfrak{T}(g) \in \mathfrak{A}^R(\Omega)$.

(iii) $\widetilde{E}_{mn}(g; S) \leqslant E^C_{mn}(g; D) \leqslant E^R_{mn}(g; D)$ *and for many "reasonable" functions g the ratio* $\rho(g) = E^R_{mn}(g; D)/\widetilde{E}_{mn}(g; S)/E^R_{mn}(g; D)$ *is very close to* 1.

In particular equality holds if g is a polynomial of degree $m+1$.

(iv) If $m \geqslant n-1$, $\widetilde{E}_{mn}(g; S) \leqslant E^C_{mn}(\mathfrak{T}(g); \Omega) \leqslant E^R_{mn}(\mathfrak{T}(g); \Omega)$.

(v) If $m \geqslant n-1$, $E^P_{mn}(\mathfrak{T}(g); \Omega) \leqslant TE^P_{mn}(g; D)$, where $P = R$ or C and $T = \|\mathfrak{T}\|$ for $m = n-1$ or $\|\mathfrak{T}_0\|$ for $m \geqslant n$, where $\mathfrak{T}_0$ is defined by $\mathfrak{T}_0(g) = \mathfrak{T}(g) + g(0)$. For convex Ω, $\|\mathfrak{T}\| \leqslant 5$ and $\|\mathfrak{T}_0\| \leqslant 2$.

Unfortunately it is not entirely clear at present what a "reasonable" function in the sense of (iii) is but it appears to have some connection with the regularity of the Taylor coefficients. (Trefethen [1981b], has some asymptotic results and some numerical computations for e^z, and further numerical results are given in Ellacott [1981]). Not every $f \in \mathfrak{A}^R(\Omega)$ is of the form $\mathfrak{T}(g)$, $g \in \mathfrak{A}^R(D)$, but for any function which can be so expressed (e.g. any polynomial or any function with a uniformly and absolutely convergent Faber series) and for $m \geqslant n-1$ (iv) and (v) yield immediately

$$\frac{E^C_{mn}(f; \Omega)}{E^R_{mn}(f; \Omega)} \geqslant \frac{\rho(\mathfrak{T}^{-1}(f))}{T}.$$

In particular,

(1a) $E^C_{n-1,n}(f; I)/E^R_{n-1,n}(f; I) \geqslant \rho(\mathfrak{T}^{-1}(f))/5$.

(1b) $E^C_{mn}(f; I)/E^R_{mn}(f; I) \geqslant \rho(\mathfrak{T}^{-1}(f))/2$, $m \geqslant n$.

If f is a polynomial of degree $m + 1$ we have

(2) $E^C_{mn}(f; I)/E^R_{mn}(f; I) \geqslant 1/2, m \geqslant n,$

and for "reasonable" functions f (i.e. functions for which $\mathfrak{T}^{-1}(f)$ is reasonable in the sense of (iii)), we would not expect the lower bounds given by (1a) and (1b) to be much less than 1/5 or 1/2 respectively.

Similar considerations hold for the problem of approximation by the real parts of rational functions as considered by Wulbert **[1978]**.

We conclude with three questions suggested by these remarks. Firstly, and most obviously, is 1/2 actually a lower bound for $E^C_{mn}(f; I)/E^R_{mn}(f; I)$ for $f \in \mathfrak{A}^R(I)$ and $m \geqslant n$, and, if so, is it sharp?

The second question is related: Are there any functions in $\mathfrak{A}^R(D)$ for which $E^C_{mn}(f; D) < E^R_{mn}(f; D)$? If so, they are likely to be more difficult to find than for I; (2) shows that $f(z) = z^2$ with $m = n = 1$ will not do. The third question is more general: Certain asymptotic results are known about the behaviour of Faber polynomials as the degree $\longrightarrow \infty$ (Pommerenke **[1964, 1967]**). Can these be applied to discuss the asymptotic behaviour of $E^C_{mn}(f; \Omega)/E^R_{mn}(f; \Omega$

REFERENCES

[1979] C. Bennet, K. Rudnick and J. Vaaler, *Best uniform approximation by linear fractional transformations*, J. Approximation Theory, **25**, 204–224.

[1981] S. Ellacott, *On the Faber transform and efficient numerical rational approximation*, Forschungsinstitut für Mathematik, ETH Zürich, SIAM J. Numer. Anal. (submitted)

[1980] D. Gaier, *Vorlesungen über Approximation im Komplexen*, Birkhäuser, Basel.

[1981] M. Gutknecht, *Rational CF approximation on a disc, on a circle and on an interval*, Seminar für Angewandte Mathematik, ETH Zürich (in preparation).

[1964] Ch. Pommerenke, *Ueber die Faberschen Polynome Schlichter Funktionen*, Math. Z. **85**, 197–208.

[1967] ———, *Konforme Abbildung und Fekete-Punkte*, Math. Z. **89**, 422–438.

[1977] A. Ruttan, *On the cardinality of a set of best complex rational approximations to a real function*, Padé and Rational Approximation (E. B. Saff and R. S. Varga, eds.), Academic Press, New York, pp. 303–319.

[1977] E. B. Saff and R. S. Varga, *Nonuniqueness of best approximating complex rational functions*, Bull. Amer. Math. Soc. **83**, 375–377.

[1978] ———, *Nonuniqueness of best complex rational approximations to real functions on real intervals*, J. Approximation Theory, **23**, 78–85.

[1981a] L. Trefethen, *Near circularity of the error curve in complex Chebyshev approximation*, J. Approximation Theory, **31**, 344–366.

[1981b] ———, *Rational Chebyshev approximation on the unit disc*, Numer. Math. **37**, 297–320.

[1979] L. Trefethen, *Near circularity of the error curve in complex Chebyshev approximation*, Computer Science Dept., Stanford Univ., J. Approximation Theory (to appear).

[1980] ———, *Rational Chebyshev approximation on the unit disc*, Stanford NAP manuscript NA-80-08, Stanford Univ., California, Numer. Math. (submitted).

[1978] D. Wulbert, *The rational approximation of real functions*, Amer. J. Math. vol. **100**, 1281–1315.

DEPARTMENT OF MATHEMATICS, BRIGHTON POLYTECHNIC, BRIGHTON, ENGLAND