29. E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.

30. R. M. Switzer, *Algebraic topology*: *Homotopy and homology*, Die Grundlehren der Math. Wissenschaften, Band 212, Springer-Verlag, New York-Heidelberg, 1975.

31. O. Veblen, *Analysis situs*, Amer. Math. Soc. Colloq. Publ., no. 5, Amer. Math. Soc., New York, 1931.

32. J. W. Vick, *Homology theory*, Pure and Applied Math., vol. 53, Academic Press, New York-London, 1973.

33. A. H. Wallace, *An introduction to algebraic topology*, Pergamon, Oxford, 1957.

34. G. W. Whitehead, *Elements of homotopy theory*, Graduate Texts in Math., no. 61, Springer-Verlag, New York-Heidelberg, 1978.

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Measurement theory: *with applications to decisionmaking*, *utility*, *and the social sciences*, by Fred S. Roberts, Encyclopedia of Mathematics and its Applications, vol. 7, Addison-Wesley, Reading, Mass., 1979, xxii + 420 pp., $24.50.

Although measurement theory is not widely known academically or within the mathematics community and claims no professional society or exclusive journal, it has developed into a cohesive body of significant proportions during the past few decades. As often happens in areas that undergo a period of intense development, measurement theory has been afforded treatment in several books, the most recent of which is the one under review. Roberts' volume was preceded by Pfanzagl's *Theory of measurement* (1968), Volume I of the *Foundations of measurement* (1971) by Krantz, Luce, Suppes and Tversky (with Volume II nearing completion), and Fishburn's more specialized *Utility theory for decision making* (1970). These earlier works are primarily technical renderings that emphasize the axiomatic approach to measurement. While Roberts also stresses axiomatics, *Measurement theory* devotes considerable space to applications.

Early work in the theory of measurement focused on empirical laws or phenomena in physics – and to a lesser extent perhaps in psychophysics, economics, and other disciplines – that could be represented numerically, and on the special properties of such representations. While empirical phenomena continue to inform and motivate the subject, recent contributions have centered on axioms for qualitative relational systems that enable mappings into numerical systems that preserve the relational structures of the qualitative systems. Major contributors include mathematicians and mathematically oriented investigators in psychology, economics, philosophy, and statistics. A significant proportion of the articles on measurement theory that have appeared in the past twenty years are in the *Journal of Mathematical Psychology*, *Econometrica*, *Psychological Review*, and the *Annals of Statistics*. Mathematically, measurement theory draws heavily upon algebra and functional analysis, and is involved in various ways with discrete mathematics, probability theory, and topology. The relatons used in its axioms are usually binary orderings, either complete or partial.

A specific theory of measurement generally consists of a set of primitives, a system of axioms on the primitives, and numerical representation and uniqueness theorems. The primitives include one or more sets, often endowed with certain structure, and one or more qualitative relations on these sets. The axioms tell how the relations order and otherwise tie together the structure; they may also postulate structural properties apart from the relations. The representation theorem specifies mappings from the qualitative system into a numerical system that preserve aspects of the qualitative relations, and the uniqueness theorem identifies the family of mappings that yield essentially the same quantitative characterization. The mappings are usually real valued, although nonstandard numbers and real vectors ordered lexicographically are sometimes used.

Ordinal, extensive, and subjective expected utility measurement illustrate how the axiomatic approach has been used in measurement theory. I will give only the flavor of the approach here: details are available in Roberts' book and the others cited earlier.

Ordinal measurement has two primitives, a set X and a binary relation $\succ$ on X, and two axioms, an ordering axiom ($\succ$ on X is a strict weak order) and an Archimedean condition postulating the existence of a countable "order-dense" subset of X. The representation theorem, which follows from a theorem of Birkhoff and Milgram, says that the axioms hold if, and only if, there is a real-valued function f on X such that, for all $x, y \in X$, $x \succ y$ iff $f(x) > f(y)$. All functions $g\colon X \to \mathrm{Re}$ that order X in the same way as f, and only these, yield a similar representation of $\succ$.

One version (Roberts, pp. 126–130) of extensive measurement uses primitives X, $\mathbf{o}$ and $\succ$, where $\mathbf{o}$ is a binary operation on X. It is assumed that $\succ$ on $(X, \mathbf{o})$ satisfies four axioms: a weak associativity condition; strict weak order for $\succ$ on X; monotonicity – $a \succ b$ iff $(a\mathbf{o}c) \succ (b\mathbf{o}c)$ iff $(c\mathbf{o}a) \succ (c\mathbf{o}b)$; and an Archimedean condition. Hölder's theorem for Archimedean ordered groups can be used to show that these axioms are sufficient (they are also necessary) for the existence of a real-valued function f on X that preserves $\succ$, as in ordinal measurement, and is additive in $\mathbf{o}$:

$$f(a\mathbf{o}b) = f(a) + f(b) \quad \text{for all } a, b \in X.$$

Extensive measurement is unique up to a similarity transformation: g also preserves $\succ$ and is additive if and only if $g = \lambda f$ for some $\lambda > 0$.

In Savage's theory of subjective expected utility (*The foundations of statistics*, 1954), $\succ$ is applied to the set X of all functions from a set S of states of the world into a set C of consequences. His axioms include the usual ordering assumption, several independence conditions that serve to separate utility u from subjective probability P, and Archimedean and dominance conditions. These axioms imply that there is a bounded function $u\colon C \to \mathrm{Re}$ and a finitely additive probability measure P on the algebra of all subsets of S such that

$$x \succ y \text{ iff } \int_S u(x(s))\, dP(s) > \int_S u(y(s))\, dP(s),$$

for all $x, y \in X$. Because of nonnecessary structural assumptions, Savage's uniqueness result is very tidy: P is unique, and u is unique up to a positive affine (linear) transformation: v serves in place of u iff there are reals $\alpha > 0$ and β such that $v = \alpha u + \beta$.

The preceding theories, plus others that involve difference measurement, product structures with additive and nonadditive representations, expected utility, subjective probability, and measurement based on partial orders and binary choice probabilities, are discussed by Roberts. Because *Measurement theory* is designed to introduce the reader to the subject without getting bogged down in mathematical details, longer proofs that are available elsewhere are not repeated. The presentation is carefully developed and is mathematically rigorous in the best sense of that phrase. At the same time, the book proceeds at a relaxed and readable pace that reflects substantial concern and expertise on the author's part to communicate with readers not previously conversant in measurement theory.

As an introduction to the axiomatic approach to measurement theory, the book succeeds well. Its value as a general introduction to measurement is considerably enhanced by numerous examples from the behavioral and social sciences. One chapter is devoted to psychophysical scaling, and there are discussions of application in energy, air-pollution, and public health.

Roberts includes a wealth of exercises that extend the theory and suggest a variety of potential applications. He has used parts of the book in an undergraduate course in mathematical models in the social sciences, and most of the book with first-year graduate students in mathematics. While I believe that *Measurement theory* is well suited for introductory courses as well as informal learning situations, it should also prove useful as a reference source for people doing research in measurement theory.

All told, I feel that Roberts' book is superbly well done, and that it should serve handsomely as *the* introduction to the theory of measurement for many years to come.

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Étale cohomology, by J. S. Milne, Princeton Univ. Press, Princeton, N.J., 1980, xiii + 323 pp., $26.50.

A journalist once asked Sir Arthur Eddington (or perhaps it was Rutherford, the story is doubtless apocryphal anyway) whether he was one of only three men in the world who understood Einstein's theory of relativity. "And who," came the reply, "is the third?"

Here is a similar story I can vouch for personally. About a week after P. Deligne proved the last of the Weil conjectures several years ago (more about these in a moment) I received through the good offices of a friend who was in France at the time some fifty pages of detailed notes on the proof. This obviously was a hot item. I was visiting a major North American university, so I offered the chairman, himself a number theorist, the notes for Xeroxing.