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VARIETIES AND UNIVERSAL MODELS IN THE THEORY OF COMBINATORIAL GEOMETRIES

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One of the most attractive but little studied ideas in the theory of combinatorial geometries (or matroids) [2], [4] is the notion of a hereditary class of geometries. A *hereditary class* of (finite) geometries is a collection of geometries which is closed under taking minors and direct sums. Thus, hereditary classes are direct analogues of varieties in universal algebra. Although varieties are highly structured collections of objects—free objects exist, for example—hereditary classes are relatively unstructured. In order to obtain reasonable results, it is necessary to impose some regularity conditions. One possibility is to postulate the existence of free (or rather, cofree) objects. A *sequence of universal models* for a hereditary class T of geometries is a sequence (T_n) of geometries in T with rank $(T_n) = n$ satisfying the universal property: if G is a geometry in T of rank n, then G is a subgeometry of T_n. A *variety* of geometries is a hereditary class with a sequence of universal models. Rather surprisingly, it is possible to classify varieties of geometries.

Before stating our result, we need to describe two simple sequences of geometries. Let M_1 be the rank one geometry and M_2 be the line with $q + 1$ points. Let $M_{2n}(q) = M_2 \oplus \cdots \oplus M_2$ and $M_{2n+1}(q) = M_2 \oplus \cdots \oplus M_2 \oplus M_1$. The subgeometries of these geometries form a variety called the *variety of matchstick geometries of order* q. Now, let B_n be the Boolean algebra on the set $\{1, \ldots, n\}$. On each of the lines $\overline{12}, \overline{23}, \ldots, \overline{(n-1)n}$, add $q - 1$ points in general position and call the resulting geometry $O_n(q)$. The subgeometries of these geometries also form a variety called the *variety of origami geometries of order q*.

THEOREM. *Let T be a variety of geometries. Then, T is one of the following collections*:

1. *the variety of free geometries*;
2. *the variety of matchstick geometries of order q*;
3. *the variety of origami geometries of order q*;
4. *the variety of all geometries coordinatizable over a fixed finite field GF(q)*;
5. *the variety of voltage-graphic geometries with voltages in a fixed finite group A*.

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The last entry merits some description. The universal models for the A-voltage-graphic geometries are the Dowling geometries $Q_n(A)$ [3]. Voltage-graphic geometries were discovered by Zaslavsky and their properties may be found in [5], [6].

We conclude with an outline of the proof. Suppose that (T_n) is the sequence of universal models for T and let q be the positive integer $|T_2| - 1$. If $q = 1$, then T_n is a Boolean algebra. Assume that $q \geqslant 2$. If T_3 is not connected, then T_n is the full matchstick geometry $M_n(q)$. We assume T_3 is connected and distinguish two cases: (a) for some n, T_n *splits*: that is to say, T_n is the union of two proper flats; (b) no T_n splits.

Case (a). Let m be the smallest index for which T_{m+1} splits. We first show that for $n \geqslant m$, T_n is the union of $n - m + 1$ flats isomorphic to T_m, and obtain a fairly precise description of how these flats are arranged in T_n. By a counting argument we show that $m = 2$. It follows that T_n is the origami geometry $O_n(q)$.

Case (b). T_3 is shown to contain three modular lines l_1, l_2, l_3 in general position. If T_3 in the union of these lines, then T_n is shown to be isomorphic to $Q_n(A)$ for some finite group A. The key idea here is to use contractions in T_4 to prove associativity in the quasigroup A defined by T_3 (see [3]). If T_3 is not the union of l_1, l_2, l_3, then we again use contractions in T_4, this time to prove that T_3 is a projective plane. An inductive argument then shows that T_n is the projective geometry $P_n(q)$ of rank n over $GF(q)$.

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