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AUTOMORPHIC FORMS AND SINGULARITIES OF COMPLEX SURFACES

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DEDICATED TO MY WIFE, BOBBIE

Suppose G is a finitely generated fuchsian group of the first kind. Let $A(k)$ be the vector space of entire automorphic forms of weight k and

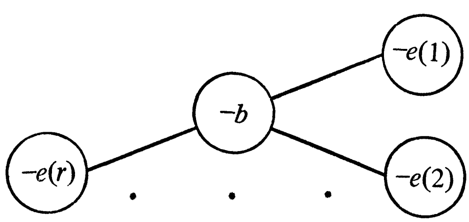
$$A(G) = \bigoplus_{k \geqslant 0} A(k)$$

the graded ring of automorphic forms. Now G acts on the upper half plane H_+ in the usual way. This action has a 'canonical' extension to $H_+ \times \mathbf{C}^*$ via

$$g(z, t) = \left(g(z), \ \frac{dg}{dz} t\right).$$

PROPOSITION 1. *$A(G)$ is a graded algebra of finite type. The algebraic set $V = \operatorname{Spec}(A(G))$ is a surface with $\mathbf{C}^*$-action. There is a Zariski open $\mathbf{C}^*$-invariant subset of V which is isomorphic to $(H_+ \times \mathbf{C}^*)/G$.*

We thus can use the theory of surfaces with $\mathbf{C}^*$-action to study the structure of the ring of automorphic forms. Now let us suppose that G is a fuchsian group with signature $\langle g; s; e(1), \ldots, e(r)\rangle$ and $V = \operatorname{Spec}(A(G))$. By [7] the singularity of V at (0) has a canonical equivariant resolution. The graph of the resolution is star shaped, of the form



where $b = 2g - 2 + r + s$.

A first step in understanding the structure of these rings is to find the minimal number of generators, n. In [9] we classified all groups with $n \leqslant 3$. The techniques there are all elementary. The results here are more general since we use

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the more powerful techniques from the theory of singularities of surfaces. The groups with $n \leqslant 3$ and $s = 0$ were also classified by Dolgacev [4]. If $g = 0$ and $s > 0$ then the singularity is a rational singularity and one can use the theory of these singularities to compute n. First we let $e = \Sigma_{i=1}^{r} e(i)$.

PROPOSITION 2. (1) *If* $s > 1$ *then* $n = e - r + s - 1$,
(2) *If* $s = 1$ *then* $n = e - 3$.

If $g = 0$ and $s = 0$ then the singularity is minimal elliptic [6], hence it follows directly that:

PROPOSITION 3. (1) *if* $r > 3$ *then* $n = \max(3, e - 8)$,
(2) *If* $r = 3$ *and* $e(i) > 2$, *for all* i, *then*

$$n = \max(3, e - 9).$$

(3) *If* $r = 3$, $e(1) = 2$, $e(2)$, $e(3) > 3$ *then*

$$n = \max(3, e - 10).$$

(4) *If* $r = 3$, $e(1) = 2$, $e(2) = 3$, $e(3) > 6$ *then*

$$n = \max(3, e - 11).$$

To get more information about generators one can use the following description of $A(k)$ as a vector space of functions on X.

PROPOSITION 4 [8], [9]. *Suppose that* $p(1), \ldots, p(r) \in X$ *are the elliptic points and* $q(1), \ldots, q(s) \in X$ *are the cusps. Then*

$$A(k) = L(kK + k(q((1) + \cdots + q(s)) + \sum_{i=1}^{r} [k(1 - 1/e(i))]p(i)$$

A major tool for stating and proving our results is the Poincaré power series of the graded algebra $A(G)$. Recall that if R is any graded algebra over a field K and M is a finitely generated R-module, then the Poincaré power series of M is defined to be

$$p(t) = \sum_{i=0}^{\infty} d(i)t^i$$

where $d(i)$ = dimension of $R(i)$ as a vector space over K. Moreover, if R is finitely generated as an algebra over K then $p(t)$ is a rational function [2]. Now let m be the maximal ideal of $A(G)$ defined by

$$m = \bigoplus_{k>0} A(k).$$

Then any basis of m/m^2 as a vector space over **C** lifts to a minimal set of generators of the algebra $A(G)$. Conversely, every minimal set of algebra generators forms a basis for m/m^2. Now m is a graded ideal, hence there is an induced grading on m/m^2. Let $p(t)$ be the Poincaré power series of m/m^2. Of course this is just a polynomial. The coefficient of t^i in this polynomial is just the number of independent generators of weight i.

THEOREM. *If $n > 3$ then*

$$p(t) = f(t) + \sum_{i=1}^{r} (t^2 + \cdots + t^{e(i)})$$

where $f(t)$ is given in the table below.

SIGNATURE	$f(t)$
$s \geqslant 3$ or $g = 0$ and $s = 2$	$(g + s - 1)t$
$s = 2, g \geqslant 2$ and $1(q(1) + q(2)) = 1$	$(g + 1)t + t^2$
$s = 2, g \geqslant 1$ and $1(q(1) + q(2)) = 2$	$(g + 1)t + gt^2$
$s = 1, g \geqslant 3$, X nonhyperelliptic	$gt + 2t^2 + t^3$
$s = 1, g \geqslant 1$, X hyperelliptic	$gt + gt^2 + t^3$
$s = 1, g = 0, r \geqslant 2$	$-t^2 + (r - 2)t^3$
$s = 0, g \geqslant 3$, X nonhyperelliptic	gt
$s = 0, g \geqslant 2$, X hyperelliptic, $g + r \geqslant 3$	$gt + (g - 2)t^2$
$s = 0, g = 1$ and either $e \geqslant 6$ or $r = 1, e(1) \geqslant 4$	$t - t^2$
$s = 0, g = 0, r \geqslant 4, e \geqslant 11$	$-3t^2 + (r - 5)t^3$
$s = 0, g = 0, r = 3, e(i) \geqslant 3$, for all i.	$-3t^2 - 2t^3 - t^4$
$s = 0, g = 0, r = 3, e(1) = 2, e(2), e(3) \geqslant 4, e \geqslant 13$	$-3t^2 - 2t^3 - t^4 - t^5$
$s = 0, g = 0, r = 3, e(1) = 2, e(2) = 3, e(3) \geqslant 9$	$-3t^2 - 2t^3 - t^4 - t^5 - t^7$

The finite number of signatures which do not appear above all have A generated by 2 or 3 elements. The generators and relations for these rings are listed in [9].

COROLLARY. *If we are as above, then $n = e - r + f(1)$.*

EMBEDDING DIMENSION 4

The following is a list of all groups whose algebra of automorphic forms is generated by four elements. If the algebra is a complete intersection, that is the ideal of relations is generated by two elements, then we give the degree of the generating relations in the last column.

SIGNATURE	DEGREES OF GENERATORS	RELATIONS
$\langle 4\rangle$, X nonhyperelliptic	1 1 1 1	2 3
$\langle 3\rangle$, X hyperelliptic	1 1 1 2	2 4
$\langle 3;0;2\rangle$, X nonhyperelliptic	1 1 1 2	3 3
$\langle 2;0;2,2\rangle$	1 1 2 2	3 4
$\langle 2;0;3\rangle$	1 1 2 3	4 4
$\langle 2;2\rangle$ $q(1)+q(2)$ not linearly equivalent to K.	1 1 1 2	3 3 4
$\langle 2;3\rangle$	1 1 1 1	2 3 3
$\langle 1;0;2,2,2,2\rangle$	1 2 2 2	4 4
$\langle 1;0;2,2,3\rangle$	1 2 2 3	4 5
$\langle 1;0;2,4\rangle$	1 2 3 4	5 6
$\langle 1;0;3,3\rangle$	1 2 3 3	4 6
$\langle 1;0;5\rangle$	1 3 4 5	6 8
$\langle 1;1;2\rangle$	1 2 2 3	4 5 6
$\langle 1;2;2\rangle$	1 1 2 2	3 4 4
$\langle 1;3;2\rangle$	1 1 1 2	3 3 3
$\langle 1;4\rangle$	1 1 1 1	2 2

The groups with $g = 0$ are easily found using Propositions 1 and 2. There are 25 signatures that occur in this case.

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