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THE RATIONAL HOMOTOPY OF FIXED POINT SETS OF TORUS ACTIONS

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1. **Introduction.** Let X be a connected topological space, whose Sullivan-de Rham minimal model, $\mathcal{M}(X)$, is finitely generated. Following Halperin [8], we shall denote the indecomposable quotient of $\mathcal{M}(X)$ by $\Pi_\psi^*(X)$, and call it the pseudo-dual rational homotopy of X. If X is simply-connected, then $\Pi_\psi^n(X)$ is naturally isomorphic to $(\pi_n(X) \otimes Q)^*$, for all $n \geqslant 1$. (See [4] and [8] for detailed treatment of $\Pi_\psi^*(X)$.)

DEFINITION 1.1. If $\dim_Q \Pi_\psi^*(X) < \infty$, then we shall say that X has finite dimensional rational homotopy (FDRH), and we shall define the Euler-Poincaré homotopy characteristic of X to be $\chi\pi(X) = \Sigma_{n=1}^\infty (-1)^n \dim_Q \Pi_\psi^n(X)$.

In this note we announce some results, which relate $\Pi_\psi^*(X)$ to $\Pi_\psi^*(F)$, where F is a component of the fixed point set of a torus group action on X. Further results and detailed proofs will appear in [2] and [3].

2. **Results.** Although more general conditions would suffice, we shall assume, for simplicity, throughout this section, that X is a compact topological manifold, that a torus T is acting on X locally smoothly (that is, with linear slices), and that the fixed point set, X^T, is nonempty. Our first theorem is the following.

THEOREM 2.1. *If X has FDRH, and if F is a component of X^T, then F has FDRH, and $\chi\pi(F) = \chi\pi(X)$. Furthermore,*

$$\text{(i)} \qquad \sum_{n=1}^{\infty} \dim_Q \Pi_\psi^{2n}(F) \leqslant \sum_{n=1}^{\infty} \dim_Q \Pi_\psi^{2n}(X);$$

and

$$\text{(ii)} \qquad \sum_{n=0}^{\infty} \dim_Q \Pi_\psi^{2n+1}(F) \leqslant \sum_{n=0}^{\infty} \dim_Q \Pi_\psi^{2n+1}(X).$$

We also have the following generalization of Bredon's inequalities [5].

THEOREM 2.2. *If X has FDRH, then, for all $n \geqslant 1$,*

$$\dim_Q \Pi_\psi^n(F) \leqslant \sum_{k=0}^{\infty} \dim_Q \Pi_\psi^{n+2k}(X).$$

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Our third theorem is a generalized Golber formula ([1], [6], [7] and [9]). We shall assume now that X has FDRH, and that $\Pi_\psi^{2n}(X) = 0$, for all $n \geqslant 1$. It follows that X^K is connected, for any subtorus $K \subseteq T$. From Theorem 2.1 it follows also that $\Pi_\psi^{2n}(X^K) = 0$, for all $n \geqslant 1$, and that X^K has FDRH. With this in mind we make the following definition.

DEFINITION 2.3. Suppose that $\Pi_\psi^*(X^K)$ has a basis (as a rational vector space) of elements with degrees $\alpha_i(K)$, $1 \leqslant i \leqslant s$.

Set

$$e(K) = \prod_{1 \leqslant i < j \leqslant s} (\alpha_i(K) + 1)(\alpha_j(K) + 1).$$

If $K = \{e\}$, so that $X^K = X$, then set $e(K) = e(X)$.

The generalized Golber formula is as follows.

THEOREM 2.4.

$$e(X) - e(T) - \sum_H [e(H) - e(T)] = \sum_K \left[e(K) - e(T) - \sum_{H \supset K} \{e(H) - e(T)\} \right],$$

where Σ_H runs over all subtori of T of corank one, Σ_K runs over all subtori of T of corank two, and $\Sigma_{H \supset K}$ runs over all subtori of T of corank one, which contain K.

In [3], we obtain further formulae of this kind, and give a general solution to Problem 9 of [9, p. 148].

3. **Method of proof.** The following theorem is the main technical device which we use.

THEOREM 3.1. *If S is a commutative overring of the rational numbers, and if $\mathcal{A}_S$ is the category of differential $(\mathbf{Z}/2\mathbf{Z})$-graded algebras over S (with S having degree 0), then $\mathcal{A}_S$ is a closed model category.*

The proof of this theorem is a straightforward analogue of the proof of Theorem 4.3 of [4].

Theorem 3.1 allows us to reproduce a localization-cum-ideal theory for Π_ψ^*, analogous to that for equivariant cohomology produced by Chang and Skjelbred [6].

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