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THE FAILURE OF SPECTRAL ANALYSIS IN L^p FOR $0 < p < 1$

BY KAREL de LEEUW[1]

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1. **Introduction.** For $0 < p < 1$, L^p is the space of measurable f on the circle group $\mathbf{T}$ with

$$\|f\|_p = \left[(2\pi)^{-1} \int_{-\pi}^{+\pi} |f(x)|^p \, dx\right]^{1/p} < \infty.$$

If $0 < p < 1$, L^p is not a Banach space, but is a metric space with distance defined by $d(f, g) = \|f - g\|_p^p$.

A linear subspace of L^p will be called a $\mathbf{T}$-*subspace* if and only if it is closed and translation invariant. If F is a function or a collection of functions in L^p, then $L^p(F)$ will denote the smallest $\mathbf{T}$-subspace of L^p containing F, the $\mathbf{T}$-subspace of L^p *generated* by F. If $F = \{e^{in\cdot}: n \in \Delta\}$, is a collection of exponential functions, $L^p(F)$ will also be denoted by $L^p(\Delta)$.

For $p \geqslant 1$, the classification of the $\mathbf{T}$-subspaces of L^p is straightforward (see [3, Chapter 11]). The map

$$\Delta \rightsquigarrow L^p(\Delta) \tag{1.1}$$

gives a 1-1 correspondence between the collection of all subsets of integers and all $\mathbf{T}$-subspaces of L^p.

The purpose of this note is to point out that the case $0 < p < 1$ is much more intricate, to be specific, the map (1.1) is neither 1-1 nor onto. We shall outline proofs of results which imply the following.

THEOREM 1. *Let* $0 < p < 1$. *Then*

(i) L^p *has nontrivial* $\mathbf{T}$-*subspaces containing no exponentials*;

(ii) *There are distinct sets* Δ *and* Γ *of integers with* $L^p(\Delta) = L^p(\Gamma)$.

Details will be published elsewhere. In what follows, "Proof" should of course be interpreted to mean "Outline of Proof".

2. **Spectral analysis in H^p for $0 < p < 1$; Cauchy integrals.** Here we restrict to the $\mathbf{T}$-subspace $L^p(\{e^{in\cdot}: n \geqslant 0\})$, which is denoted by H^p. (For the basic properties of H^p which we use in what follows, see [2, Chapter 7], [4, Chapter 3] or [1].) H^p can also be characterized as follows: Let $\mathbf{D}$ be the unit disk $\{z: |z| < 1\}$. We define $H^p(\mathbf{D})$ to consist of all functions F which are analytic in D with $|||F|||_p = \sup\{\|F_r\|_p: 0 < r < 1\} < \infty$, where each F_r is de-

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fined on **T** by $F_r(e^{i\theta}) = F(re^{i\theta})$. The functions in $H^p(\mathbf{D})$ have boundary values a.e. on **T** and the mapping $F \rightsquigarrow \widetilde{F}$ defined by $\widetilde{F}(e^{i\theta}) = \lim_{r\to 1} F(re^{i\theta})$, a.e. $e^{i\theta} \in \mathbf{T}$, is an isometry of $H^p(\mathbf{D})$ onto H^p.

We shall denote by L_0^p the **T**-subspace $H^p \cap \overline{H^p} = \{ f: f$ and $\overline{f}$ are in $H^p\}$. Propositions 2.2 and 2.3 below show that L_0^p is quite large if $p < 1$ even though it consists only of constant functions if $p \geqslant 1$. We first indicate how L_0^p is our "universal counterexample" to spectral analysis.

PROPOSITION 2.1. *L_0^p contains no nonconstant exponential functions.*

PROOF. We may assume $p < 1$. If $H^p \cap \overline{H^p}$ contained a nonconstant exponential function, it would contain some $e^{in\cdot}$ for $n < 0$. By Theorem 7.35 of [5], $H^p \cap L^1 = H^1$. But $e^{in\cdot} \notin H^1$.

The above proof of course yields a great ideal more than asserted by Proposition 2.1, namely, that $L_0^p \cap L^1$ consists only of constant functions.

If μ is a finite Borel measure on **T**, we define F_μ by

$$F_\mu(z) = \int \frac{w}{w - z}\, d\mu(w), \qquad |z| \neq 1. \tag{2.1}$$

The restriction of F_μ to the unit disk **D** will be denoted by C_μ. By Theorem 3.5 of [1], C_μ is in each $H^p(\mathbf{D})$ for $p < 1$ and thus its boundary function $\widetilde{C}_\mu$ is in each H^p for $p < 1$. We will call $\widetilde{C}_\mu$ the *Cauchy transform* of μ.

PROPOSITION 2.2. *Let $0 < p < 1$. Then L_0^p contains the Cauchy transforms of all singular measures on* **T**.

PROOF. Let μ be a singular measure on **T**. Define F_μ by (2.1) so $C_\mu = F_\mu$ on **D**. Then, for $z = re^{i\theta}$, $|z| < 1$, $F(z) - F(1/\overline{z}) = \Sigma_{-\infty}^{+\infty} \hat{\mu}(n) r^{|n|} e^{in\theta}$, which is the rth Abel mean of the Fourier series of μ. Since μ is singular, Theorem 1.2 of [1] shows that the series converges a.e. in **T** to 0. Thus, the function G defined in **D** by $G(z) = \overline{F(1/\overline{z})}$ has boundary values conjugate to $\widetilde{C}_\mu$ a.e. on **T**. It remains to show that $G \in H^p(\mathbf{D})$ for each $p < 1$. Since $G(z) = -\Sigma_{n=1}^{\infty} \hat{\mu}(-n) z^n$ in **D**, if the measure η is defined on **T** by $\eta(E) = -\int_E w\,d\mu(-w)$, $G = C_\eta$ and thus $G \in H^p(\mathbf{D})$ for each $p < 1$ by Theorem 3.5 of [1].

We can assert a converse to Proposition 2.2. It is an easy consequence of (i) of Theorem 2.4 below that if $p < 1$, and μ is a measure on **T**, $\widetilde{C}_\mu \in L_0^p$ if and only if μ is a singular measure plus a constant multiple of Lebesgue measure.

A bounded analytic function ϕ defined in **D** is called *inner* if $|\phi(z)| < 1$, $z \in \mathbf{D}$, and $|\widetilde{\phi}(e^{i\theta})| = 1$, a.e. $e^{i\theta} \in \mathbf{T}$.

PROPOSITION 2.3. *Let $f \in L_0^p$. If ϕ is inner with $\phi(0) = 0$, then $f \circ \widetilde{\phi} \in L_0^p$.*

PROOF. Let $f \in L_0^p$. Then there are G and K in $H^p(\mathbf{D})$ with $\widetilde{G} = f$ and $\widetilde{H} = \overline{f}$ in L^p. That $G \circ \phi$ and $K \circ \phi$ are in $H^p(\mathbf{D})$ follows from Theorem 1.7 of [1]. Let X be the set of $e^{i\theta} \in \mathbf{T}$ where $\lim_{r\to 1} H(re^{i\theta}) = \lim_{r\to 1} G(re^{i\theta})$. Since

X has measure 2π, $\tilde{\phi}(e^{i\theta})$ must be in X for a.e. $e^{i\theta} \in \mathbf{T}$. Thus $(G \circ \phi)^\sim = f \circ \tilde{\phi}$ and $(H \circ \phi)^\sim = \bar{f} \circ \tilde{\phi} = (f \circ \tilde{\phi})^-$, which shows that $f \circ \tilde{\phi} \in L_0^p$.

One more definition before we state the main result of this section. If $F(z) = \Sigma_{n=0}^\infty a_n z^n$ is analytic in $\mathbf{D}$, we define spec F to be $\{n: a_n \neq 0\}$.

THEOREM 2.4. *Let $0 < p < 1$. Suppose that μ is a finite Borel measure on* $\mathbf{T}$.

(i) *If μ is absolutely continuous, then $L^p(\tilde{C}_\mu) = L^p(\text{spec } C_\mu)$.*

(ii) *If μ is singular, then $L^p(\tilde{C}_\mu)$ contains no nonconstant exponential functions.*

PROOF. (The equality is clear if $\tilde{C} \in L^1$. But we only have that $\tilde{C}_\mu \in L^r$ for each $r < 1$.) $\tilde{C}_\mu \in L^p(\text{spec } C_\mu)$ since the Fourier series of $\tilde{C}_\mu$ is Abel summable to $\tilde{C}_\mu$ in $\| \cdot \|_p$ (see p. 284 of [5]). Thus, $L^p(\tilde{C}_\mu) \subseteq L^p(\text{spec } C_\mu)$. That $L^p(\text{spec } C_\mu) \subseteq L^p(\tilde{C}_\mu)$ follows by an appropriate adaptation of the discussion on p. 263 of [5]. (ii) follows from Proposition 2.1 and 2.2.

To see that (i) of Theorem 1 is a consequence of Theorem 2.4, take $L^p(\tilde{C}_\mu)$, where μ is any singular measure on $\mathbf{T}$ with $\int d\mu = 0$.

Theorem 2.4 lends weight to the following conjecture: If λ is a measure on $\mathbf{T}$ with absolutely continuous part μ, then $L^p(\tilde{C}_\lambda)$ and $L^p(\tilde{C}_\mu)$ contain the same exponentials if $p < 1$.

There are other natural topologies besides the norm topology for H^p in the case $0 < p < 1$, in particular, the weak topology and the topology induced by the containing space in the sense of [2]. Routine arguments show that in these topologies the $\mathbf{T}$-invariant subspaces of H^p are in 1-1 correspondence with the subsets of the nonnegative integers, as is the case when $1 \leqslant p < \infty$ and H^p has the norm topology.

3. **Distinct sets of exponentials spanning the same subspace of L^p.** If μ is a finite Borel measure on $\mathbf{T}$, its spectrum is $\{n: \hat{\mu}(n) \neq 0\}$. The following implies (ii) of Theorem 1.

THEOREM 3.1. *Let $0 < p < 1$. Suppose that Γ is the spectrum of a singular measure on $\mathbf{T}$ and that Δ is obtained from Γ by deleting a finite number of elements. Then $L^p(\Gamma) = L^p(\Delta)$.*

PROOF. For $0 < r < 1$, define F_r on $\mathbf{T}$ by $F_r(e^{i\theta}) = \Sigma_{-\infty}^{+\infty} \hat{\mu}(n) r^{|n|} e^{in\theta}$. Since F_r is the rth Abel mean of the Fourier series of μ and μ is singular, Theorem 1.2 of [1] shows that $\lim_{r \to 1} F_r(e^{i\theta}) = 0$, a.e. $e^{i\theta} \in \mathbf{T}$. $\{F_r: 0 < r < 1\}$ is bounded in L^1 and thus $\lim_{r\to 1} \|F_r\|_p = 0$. Let $\Lambda = \{n: n \in \Gamma, n \notin \Delta\}$ and define the trigonometric polynomial P on $\mathbf{T}$ by

$$P(e^{i\theta}) = - \sum_{n \in \Lambda} \hat{\mu}(n) e^{in\theta},$$

so

$$\lim_{r\to 1}\left\| \sum_{n\in\Delta} \hat{\mu}(n) r^{|n|} e^{in\cdot} - P \right\|_p = 0.$$

Thus $P \in L^p(\Delta)$, and as a consequence each $e^{im\cdot}$ with $m \in \Lambda$ is in $L^p(\Delta)$, so $L^p(\Gamma) = L^p(\Delta)$.

BIBLIOGRAPHY

1. P. Duren, *Theory of H^p spaces*, Pure and Appl. Math., vol. 38, Academic Press, New York, 1970. MR 42 #3552.

2. P. Duren, B. Romberg and A. Shields, *Linear functionals on H^p spaces with $0 < p < 1$*, J. Reine Angew. Math. 238 (1969), 32–60. MR 41 #4217.

3. R. E. Edwards, *Fourier series: A modern introduction.* Vols. 1, 2, Holt, Rinehart and Winston, New York, 1967. MR 33 #4659; 35 #7062.

4. Y. Katznelson, *An introduction to harmonic analysis*, Wiley, New York, 1968. MR 40 #1734.

5. A. Zygmund, *Trigonometric series*, Vols. 1, 2, 2nd ed., Cambridge Univ. Press, New York, 1959. MR 21 #6498.

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, STANFORD, CALIFORNIA 94305