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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MARYLAND 20742

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A COHOMOLOGICAL STRUCTURAL THEOREM FOR TOPOLOGICAL ACTIONS OF Z_2-TORI ON SPACES OF Z_2-COHOMOLOGY TYPE OF SUCCESSIVE FIBRATION OF PROJECTIVE SPACES

BY WU-YI HSIANG[1] AND J. C. SU

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Let X be a given G-space and $X \to X_G \to B_G$ be the universal bundle with X as its typical fibre. We shall consider the *ordinary cohomology of the total space* $H^*(X_G)$ as the *equivariant cohomology of* X, namely, we shall take $H^*_G(X) = H^*(X_G)$ as the definition of the equivariant cohomology theory. In case G are *elementary abelian groups* (i.e., tori or Z_p-tori), several fundamental cohomological splitting theorems are formulated and proved in [1], [2] which establish definitive, neat correlations between the *cohomological orbit structures* (e.g., $H^*(F)$, orbit types, etc.) of the given G-space X and the various *ideal theoretical invariants of* $H^*_G(X)$. In the simplest cases that $H^*(X)$ are generated by a single generator (e.g., spheres, projective spaces), the ideals occur in such cohomological splitting theorems are automatically principal ideals. Therefore the cohomological structural theorems for topological ac-

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.ions of elementary abelian groups on such spaces are immediate, direct consequences of the fundamental splitting theorems (cf. [2, Chapters IV, V, VI]) Since this kind of cohomological structural theorems plays a similar role in the study of *topological* transformation groups as that of the Schur lemma in the study of *linear* transformation groups, one may consider such cohomological structural theorems as a kind of *topological* Schur lemma.

In applying the fundamental cohomological splitting theorems to establish the same kind of topological Schur lemma for spaces X with more complicated cohomology structures $H^*(X)$, the ideal theoretical equations become a set of formidable algebraic problems. In order to develop sufficient understanding towards an eventual successful solution of such *geometrically originated ideal theoretical equations*. it is rather natural to begin our general investigation with some suitable *testing spaces.* The purpose of this note is to announce a theorem which represents a beginning success of such an effort.

Let $X = X_k \longrightarrow X_{k-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0$ be a sequence of fibrations such that (i) X_0 is a point and (ii) $X_i \longrightarrow X_{i-1}$ is the associated projective bundle of a real vector bundle $E_i \longrightarrow X_{i-1}$. Cohomologically, it is well known that

$$H^*(X_i; \mathbf{Z}_2) \simeq H^*(X_{i-1}; \mathbf{Z}_2)[x_i] / \langle f_i \rangle$$

where

$$f_i = x_i^{n_i} + w_1 x_i^{n_i - 1} + \cdots w_l x_i^{n_i - l} + \cdots w_{n_i}$$

and w_l are the lth Whitney class of $E_i \longrightarrow X_{i-1}$. Therefore, it is clear that

$$H^*(X; \mathbf{Z}_2) \simeq \mathbf{Z}_2 [x_1, x_2, \ldots, x_k] / \langle f_1, f_2, \ldots, f_k \rangle$$

where $\deg x_i = 1$ and $f_i \in \mathbf{Z}_2 [x_1, \ldots, x_i]$ with $x_i^{n_i}$ as its leading term. We shall use spaces of the above $\mathbf{Z}_2$-cohomology type as our testing spaces. The formulation of the cohomological structural theorem for topological actions of $\mathbf{Z}_2$-tori on spaces of above $\mathbf{Z}_2$-cohomology type is strongly influenced by the following simple-minded "linear models":

Suppose G is a $\mathbf{Z}_2$-tori and $E_i \longrightarrow X_{i-1}$ are *equivariant* G-bundles, $1 \leqslant i \leqslant k$. Then all X_i are G-spaces. It is not difficult to see that the cohomological orbit structures of such a G-space X_k are completely determined by the following family of $\mathbf{Z}_2$-weight systems:

(i) Let $\Omega_0 = \{\alpha_{j_1}; k_{j_1}\}$ be the weight system of the linear G-action on the fibre over $X_0 = \{pt\}$. Then the fixed point set $F(G, X_1)$ consists of connected components naturally indexed by the *distinct* weights of Ω_1, i.e., $F(G, X_1) = \Sigma F_{j_1}, F_{j_1} = RP^{(k_{j_1} - 1)}$.

(ii) Let $q_{j_1} \in F_{j_1}$ and $\Omega_{j_1} = \{\alpha_{j_1j_2}; k_{j_1j_2}\}$ be the weight system of the linear G-action on the fibre over q_{j_1}. Then $F(G, X_2) = \Sigma F_{j_1j_2}$ where $F_{j_1j_2} \longrightarrow F_{j_1}$ are fibrations of projective spaces.

(iii) In general, let $q_{j_1 \ldots j_{(i-1)}} \in F_{j_1 \ldots j_{(i-1)}}$ and $\Omega_{j_1 \ldots j_{(i-1)}} = \{\alpha_{j_1 \ldots j_i}; k_{j_1 \ldots j_i}\}$ be the weight system of the linear G-action on the fibre over $q_{j_1 \ldots j_{(i-1)}}$. Then $F(G, X_i) = \Sigma F_{j_1 \ldots j_i}$, where $F_{j_1 \ldots j_i} \longrightarrow F_{j_1 \ldots j_{(i-1)}} \longrightarrow \cdots \longrightarrow F_{j_1j_2} \longrightarrow F_{j_1}$ are fibrations of projective spaces.

MAIN THEOREM. *Let G be a $\mathbf{Z}_2$-torus and X be a G-space of $\mathbf{Z}_2$-cohomology type of successive fibrations of real projective spaces. Assume that $F(G, X) \neq \emptyset$, the local coefficient system in $\mathbf{Z}_2$-Serre spectral sequence of $X_G \longrightarrow B_G$ is trivial, and assume, moreover, the following dimension restriction, namely, $n_1 \leqslant 2^{a_1} < n_2 \leqslant 2^{a_2} < \cdots \leqslant 2^{a(k-1)} < n_k$. Then* (i) *$X$ is totally nonhomologous to zero, so that $H_G^*(X; \mathbf{Z}_2) \simeq R \times_{\mathbf{Z}_2} H^*(X; \mathbf{Z}_2)$ as an R-module, where $R = H^*(B_G; \mathbf{Z}_2)$.*

(ii) *Let $\{x_i, 1 \leqslant i \leqslant k\}$ be liftings of the generators $\overline{x_i} \in H^*(X)$ into $H_G^*(X)$. Then*

$$H_G^*(X) \simeq R[x_1, \ldots, x_k]/\langle g_1, \ldots, g_k\rangle$$

where g_i is a polynomial in $R[x_1, \ldots, x_i]$ (i.e., g_i is independent of $x_{i+1}, \ldots, x_k$) with $x_i^{n_i}$ as its leading term, and moreover, there exist $\alpha_{j_1}, \alpha_{j_1j_2}, \ldots, \alpha_{j_1 \ldots j_k} \in H^1(B_G)$ such that

$$g_1(x_1) = \prod_{j_1} (x_1 + \alpha_j)^{k_{j_1}},$$

$$g_2(\alpha_{j_1}, x_2) = \prod_{j_2} (x_2 + \alpha_{j_1j_2})^{k_{j_1j_2}}, \ldots,$$

$$g_i(\alpha_{j_1}, \ldots, \alpha_{j_1 \ldots j_{(i-1)}}, x_i) = \prod_{j_i} (x_i + \alpha_{j_1 \ldots j_i})^{k_{j_1 \ldots j_i}}, \quad 1 \leqslant i \leqslant k.$$

(iii) *There is a natural bijection between the connected components of $F(G, X)$ and the k-tuples of "roots" $(\alpha_{j_1}, \alpha_{j_1j_2}, \ldots, \alpha_{j_1 \ldots j_k})$. Each component $F_{j_1 \ldots j_k}$ is again of the $\mathbf{Z}_2$-cohomology type of k-fold fibration of real projective spaces, namely*

$$H^*(F_{j_1 \ldots j_k}) \simeq \mathbf{Z}_2[y_1, \ldots, y_k]/\langle h_{j_1}, h_{j_1j_2}, \ldots, h_{j_1 \ldots j_k}\rangle$$

where $h_{j_1 \ldots j_i}$ is a homogeneous polynomial in $\mathbf{Z}_2[y_1, \ldots, y_i]$ with $y^{k_{j_1 \ldots j_i}}$ as its leading term, and moreover, the polynomials $\{g_i\}$ and $\{h_{j_1 \ldots j_i}\}$ are related as follows:

$$g_{i+1} \equiv \prod_{j(i+1)} h_{j_1 \ldots j(i+1)}(x_1 + \alpha_{j_1}, x_2 + \alpha_{j_1 j_2}, \ldots, x_{i+1} + \alpha_{j_1 \ldots j_{(i+1)}})$$
$$\text{modulo } (h_{j_1}(x_1 + \alpha_{j_1}), \ldots, h_{j_1 \ldots j_i}(x_1 + \alpha_{j_1}, \ldots, x_i + \alpha_{j_1 \ldots j_i})).$$

(iv) *The $\mathbf{Z}_2$-cohomological orbit structures of X are the same as that of the "linear model" with*

$$\Omega_0 = \{\alpha_{j_1}; k_{j_1}\}, \ldots, \Omega_{j_1 \ldots j_{(i-1)}} = \{\alpha_{j_1 \ldots j_i}; k_{j_1 \ldots j_i}\}, \ldots, \qquad 1 \leqslant i \leqslant k.$$

Therefore, $\{\Omega_0, \Omega_{j_1}, \ldots, \Omega_{j_1 \ldots j_i}, \ldots, \Omega_{j_1 \ldots j_{(k-1)}}\}$ *is called the family of geometric weight systems of X.*

In the assumptions of the above Theorem, the undesirable part is the dimension restriction. This is needed to simplify some computations of Steenrod square operations. We believe that it can be removed by more elaborate computations. Once such dimension restrictions can be removed, then such a structural theorem for $\mathbf{Z}_2$-tori can be used to deduce a similar structural theorem for tori which can then be served as the topological Schur lemma in the study of topological actions of compact connected Lie groups on this type of spaces. Similar theorems hold for spaces of $\mathbf{Z}_2$-cohomology type of successive fibrations of projective spaces of mixed type (i.e., deg x_i = 1, 2, or 4). Such theorems and their applications in the study of topological actions of general compact Lie groups will be appearing in a later paper.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MASSACHUSETTS, AMHERST, MASSACHUSETTS 01002