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ON LOCAL SOLVABILITY OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS NOT OF PRINCIPAL TYPE

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I. **Introduction.** Necessary and sufficient conditions have been found [5], [6], [7] for the local solvability of linear partial differential operators of principal type. An operator $P(x, D)$ of order m on an open domain $\Omega \subset \mathbf{R}^N$ is said to be of principal type in Ω if $P_m(x, \xi) = 0, x \in \Omega$, $\xi \in \mathbf{R}^N \sim \{0\}$ implies that $\nabla_\xi P_m(x, \xi) \neq 0$. L. Nirenberg and F. Treves [6], [7] have shown that if

(i) $P_m(x, D)$ has analytic coefficients,

(ii) for all complex numbers z, $\operatorname{Im}(zP_m)$ does not change signs in Ω along any null-bicharacteristic strip of $\operatorname{Re}(zP_m)$,

then $P(x, D)$ is locally solvable in Ω. Hereafter we shall say that an operator $P(x, D)$ of principal type satisfies the N-T (Nirenberg-Treves) condition if $P_m(x, D)$ satisfies conditions (i) and (ii) above.

We remark here that for operators of principal type, local solvability depends only upon the leading terms. By contrast, for operators not of principal type one must consider lower order terms. Similar considerations arise in determining the hyperbolicity of operators with multiple real roots in their principal parts [1], [3], [4].

With the above remarks in mind, we specify our problem. Let $P_m(x, D) = Q_1^{j_1}(x, D) \circ \cdots \circ Q_k^{j_k}(x, D)$ with

(i) Each $Q_i(x, D)$ a homogeneous operator of principal type which satisfies the N-T condition.

(ii) $Q_i^{j_i}(x, D) = Q_i(x, D) \circ \cdots \circ Q_i(x, D)$ j_i-times.

(iii) The $Q_i(x, \xi)$'s having no common real roots except for $\xi = 0$.

We state the following definition.

DEFINITION. An operator $T(x, D)$ of order $l < m$ is an admissible lower order perturbation of $P_m(x, D)$ if $\forall b \in C^\infty(\Omega), P_m + bT$ is locally solvable in Ω.

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In this note we state a sufficient condition for admissibility, and in the case where $\max\{j_i\}= 2$, a weaker necessary condition. The proofs will be published elsewhere.

II. **Sufficiency**. Let $P = P_m + bT$, and define the operator $P_j(x, D)$ by $P_j(x, D) = Q_1{}^{[j_1-m+j]}(x, D) \circ \cdots \circ Q_k{}^{[j_k-m+j]}(x, D)$, where $[s] = \max\{s, 0\}$. We assume that

$$T = R_l \circ P_l + R_{l-1} \circ P_{l-1} + \cdots + R_0 \circ P_0$$

with order $(R_j \circ P_j) \leqslant j$. This immediately implies that

$$P = P_m + R_{m-1} \circ P_{m-1} + \cdots + R_0 \circ P_0.$$

Many applications of the fact that for any two differential operators $L(x, D)$ and $H(x, D)$, the commutator $L(x, D) \circ H(x, D) - L(x, D) \circ H(x, D)$ is of order $\leqslant$ order(L) + order(H) $-$ 1, enables us to state that if $\widetilde{P}_j(x, D)$ has precisely the same factors as $P_j(x, D)$, then there exist operators $\widetilde{R}_j(x, D)$ such that

$$P = \widetilde{P}_m + \widetilde{R}_{m-1} \circ \widetilde{P}_{m-1} + \cdots + \widetilde{R}_0 \circ \widetilde{P}_0.$$

We can then assume that $j_1 \geqslant j_2 \geqslant \cdots \geqslant j_k$, and we define the operators $S_l(x, D)$ by $S_l = Q_1 \circ \cdots \circ Q_{j_l}, j_l = \max\{j_i : j_i - l \geqslant 0\}$, $l = 1, 2, \cdots, r = \max\{j_i\}$. One notes that since the $Q_i(x, \xi)$'s have no common real roots, $S_l(x, D)$ satisfies the N-T condition. Then since the formal adjoint $P^{\#}(x, D)$ has the same form as $P(x, D)$,

$$P^{\#} = S_1^{\#} \circ \cdots \circ S_r^{\#} + R_1 \circ S_2^{\#} \circ \cdots \circ S_r^{\#} + \cdots + R_r$$

for appropriate differential operators $R_j(x, D)$. We now define the $|||\cdot|||_{p+s}$ norm by

$$|||\varphi|||_{p+s} = \|\Lambda^s(D) \circ S_2^{\#}(x, D) \circ \cdots \circ S_r^{\#}(x, D)\varphi\|_{m_1-1} \quad \forall \varphi \in C_0^\infty(\Omega)$$

where m_1 = order $(S_1^{\#}(x, D))$. The following theorem is crucial.

THEOREM I. *Let $x_0 \in \Omega$. Then for all real numbers s, there exists a neighborhood $\omega = \omega(s)$ of x_0 and a constant $C = C(s)$ such that*

$$\|P^{\#}(x, D)\varphi\|_s \geqslant C|||\varphi|||_{p+s} \quad \forall \varphi \in C_0^\infty(\omega).$$

Define $H_{p+s}(\omega)$ to be the completion of $C_0^\infty(\omega)$ under the $|||\cdot|||_{p+s}$ norm, and $H_{-p-s}(\omega)$ to be the completion of $C_0^\infty(\omega)$ under the norm

$$|||\varphi|||_{-p-s} = \sup_{\psi \in C_0^\infty(\omega)} \frac{|\langle\varphi, \psi\rangle_{L_2(\omega)}|}{|||\psi|||_{p+s}}.$$

A standard argument then shows that $\forall f \in H_{-p-s}(\omega)$, $\exists\, u \in H_{-s}(\mathbf{R}^N)$ such that $P(x, D)u = f$ in ω.

III. **Necessity.** We have the following result on necessity when $\max\{j_i\} = 2$.

THEOREM II. *Let $P_{m-1}(x, D)$ be an admissible $m - 1$st order perturbation of $P_m(x, D)$. Then $\forall\, x_0 \in \Omega$, ξ^0 a root of $Q_1^{j_1}(x_0, \xi) \cdots Q_k^{j_k}(x_0, \xi)$ implies that ξ^0 is a root of the principal part, $\widetilde{P}_{m-1}(x_0, \xi)$, of $P_{m-1}(x_0, \xi)$.*

We may assume that $x_0 = 0$. Standard arguments [2] then reduce the proof to showing that for arbitrary integers K and L, $\exists\, b \in C^\infty(\Omega)$ such that

$$\overline{\lim_{\lambda\to\infty}} \sum_{|\beta| \leqslant L} \sup|D^{\beta\,t}(P_m + bP_{m-1})v_\lambda| < \infty$$

where

$$v_\lambda = \lambda^{N+K+1} \sum_{\mu=0}^{r-1} \exp(i(\lambda^2 u(x) + \lambda\psi(x)))\varphi_\mu \lambda^{-\mu}$$

with

(i) $\varphi_\mu \in C_0^\infty(\omega)$, u, $\psi \in C^\infty(\omega)$, ω an open neighborhood of 0.

(ii) $\nabla u(0) = \xi^0$, $\operatorname{Im} u(x) \geqslant 0$, $x \in \omega$.

(iii) $\operatorname{Im} \psi(x) \geqslant a|x|^2$, $a > 0$, $x \in \omega$.

(iv)
$$[{}^t(P_m + bP_{m-1})\exp(i(\lambda^2 u(x) + \lambda\psi(x)))]\,\exp(-i(\lambda^2 u(x) + \lambda\psi(x)))$$
$$= O(|x|^{2r})\lambda^{2(m-1)} + \text{lower order terms in } \lambda,$$
$$\text{with } r = N + K + 1 + 2(m-1) + 2L.$$

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