BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 3, May 1974

ON REARRANGEMENTS OF WALSH-FOURIER SERIES AND HARDY-LITTLEWOOD TYPE MAXIMAL INEQUALITIES[1]

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Communicated by Alberto Calderón, September 22, 1973

ABSTRACT. In this note we study the a.e. convergence properties of certain rearrangements of the Walsh-Fourier series, and maximal functions of the Hardy-Littlewood type that arise from these rearrangements.

The rearrangements are defined as follows. Let r_n be the nth Rademacher function. For $N=1, 2, \cdots$, let σ_N be a permutation of the nonnegative integers such that $\sigma_N(j)=j$ for all $j \geqq N$. If $2^N \leqq n < 2^{N+1}$, $n=\sum_{j=0}^{N} \varepsilon_j 2^j$, where $\varepsilon_j=0$ or 1 if $0 \leqq j \leqq N-1$, and $\varepsilon_N=1$, we define

$$\phi_n = \prod_{j=0}^{N} r_{\sigma_N(j)}^{\varepsilon_j}.$$

We also define $\phi_0=1$ and $\phi_1=r_0$.

If σ_N is the identity permutation, $N=1, 2, \cdots$, we recover the Walsh system. If $\sigma_N(j)=N-j-1$, $0 \leqq j \leqq N-1$, $\{\phi_n\}$ is the Walsh-Kaczmarz system. (See [**1**], [**8**] and [**12**].) In general, the system $\{\phi_n\}$ is a rearrangement of the Walsh system within dyadic blocks of indices $2^N \leqq n < 2^{N+1}$, $N=1, 2, \cdots$.

We have the following result on the a.e. convergence of Fourier series with respect to $\{\phi_n\}$. For $f \in L^1(0, 1)$, let $S_n f = \sum_{j=0}^{n-1} \phi_j \int_0^1 f\phi_j \, dt$ denote the nth partial sum of the Fourier series of f with respect to $\{\phi_n\}$, and $Mf = \sup_n |S_n f|$.

THEOREM 1. *There are absolute constants C and C_p such that*
(a) $\|Mf\|_p \leqq C_p \|f\|_p$, $f \in L^p$, $2 \leqq p < \infty$.
(b) $m\{Mf > y\} \leqq C \exp(-Cy/\|f\|_\infty)$, $y > 0$, $f \in L^\infty$.

This implies the a.e. convergence of $S_n f$ to f for $f \in L^p$, $2 \leqq p < \infty$.

If we restrict ourselves to a subclass of rearrangements, we obtain better a.e. convergence results. We say that the permutations $\{\sigma_N\}$ satisfy the

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AMS (MOS) subject classifications (1970). Primary 42A56; Secondary 42A20, 46E30.

[1] This work is part of a doctoral dissertation written under the direction of Professor Richard A. Hunt at Purdue University.

"block condition" if for each $N=1, 2, \cdots, 0 \leqq m \leqq N-1$, there is an integer $k_{N,m}$, with $0 \leqq k_{N,m} \leqq N-m-1$, such that

$$(1) \qquad \{\sigma_N(0), \cdots, \sigma_N(m)\} = \{k_{N,m}, k_{N,m}+1, \cdots, k_{N,m}+m\}.$$

THEOREM 2. *If $\{\sigma_N\}$ satisfies the block condition, then there are absolute constants C and C_p such that*

(a) $\|mf\|_p \leqq C_p \|f\|_p, f \in L^p, 1<p<2.$

(b) $\|Mf\|_1 \leqq C \int_0^1 |f|(\log^+|f|)^3\, dx + C, f \in L(\log^+ L)^3.$

(c) *If* $\int_0^1 |f|(\log^+|f|)^2 \log^+\log^+|f|\, dx < \infty$, *then* $S_n f$ *converges to* f *a.e.*

The absolute constants C and C_p in the above theorems are independent of the permutations $\{\sigma_N\}$.

The proofs of these theorems involve a modification of the Carleson-Hunt technique (see [3], [6] and [7]), and L^p boundedness of certain maximal functions of the Hardy-Littlewood type. We will only give the proofs of the estimates of the maximal functions. Complete proofs of these theorems are contained in [11]. They will appear elsewhere in the Vilenkin group setting in a joint paper with J. Gosselin [5].

To prove Theorem 2, we will show that the maximal operator

$$f \to f^* = \sup_{0 \leqq m < N; N} E(|f| \mid r_{\sigma_N(0)}, \cdots, r_{\sigma_N(m)})$$

is of weak type (p, p) $(p>1)$. Note that for the case where σ_N is the identity permutation, $N=1, 2, \cdots$, this operator is just the usual dyadic Hardy-Littlewood operator.

LEMMA 1. *If $\{\sigma_N\}$ satisfies the block condition, then for $1<p<\infty$,*

$$m\{f^* > y\} \leqq C_p^p y^{-p} \int_0^1 |f|^p\, dx,$$

where $y>0, f \in L^p$, and $C_p \leqq p/(p-1)$.

In view of (1), this is a corollary of

LEMMA 2. *For $1<p<\infty$,*

$$m\left\{\sup_{m,n} E(|f| \mid r_n, \cdots, r_{n+m}) > y\right\} \leqq C_p^p y^{-p} \int_0^1 |f|^p\, dx,$$

where $y>0, f \in L^p$, and $C_p \leqq p/(p-1)$.

PROOF. We observe that for any L^1 function g and integers $n, m \geqq 0$

$$\begin{aligned} E(g \mid r_n, \cdots, r_{n+m}) &= E(E(g \mid r_0, \cdots, r_{n+m}) \mid r_n, \cdots, r_{n+m}) \\ &= E(E(g \mid r_0, \cdots, r_{n+m}) \mid r_n, r_{n+1}, \cdots). \end{aligned}$$

The last inequality follows from the independence of the Borel fields $\mathscr{F}(r_0, \cdots, r_{n+m})$ and $\mathscr{F}(r_{n+m+1}, r_{n+m+2}, \cdots)$. (See, for example, [**4**, p. 285].) Therefore

$$
\begin{aligned}
m&\left\{\sup_{m,n} E(|f| \mid r_n, \cdots, r_{n+m}) > y\right\} \\
&\leqq m\left\{\sup_{n} E\left(\sup_{k} E(|f| \mid r_0, \cdots, r_k) \mid r_n, r_{n+1}, \cdots\right) > y\right\} \\
&\leqq y^{-p} \int_0^1 \sup_k |E(|f| \mid r_0, \cdots, r_k)|^p \, dx \\
&\leqq C_p^p y^{-p} \int_0^1 |f|^p \, dx,
\end{aligned}
$$

where $C_p \leqq p/(p-1)$. Here we have used Doob's inequality [**10**, p. 91]. This completes the proof of Lemma 2.

REMARKS. It is interesting to note that the mapping

$$f \rightarrow \sup_{m,n} E(|f| \mid r_n, \cdots, r_{n+m})$$

is not of weak type $(1, 1)$. This accounts for the fact that the argument we use only enables us to establish the a.e. convergence result for the rearranged series for functions in the class $L(\log^+ L)^2 \log^+ \log^+ L$, whereas, for the Walsh-Fourier series, a similar argument yields the same result for functions in the class $L(\log^+ L)\log^+ \log^+ L$. (See [**9**].)

The following is an example of K. H. Moon. We will construct a sequence of functions $\{g_k\}$, $0 \leqq g_k \in L^1$, such that

$$m\left\{\sup_{n,m} E(g_k \mid r_n, \cdots, r_{n+m}) > \frac{1}{2}\right\} \geqq \frac{1}{2}, \qquad k = 1, 2, \cdots,$$

but

$$\int_0^1 |g_k| \, dx \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

For each $k = 1, 2, \cdots, j = 0, 1, \cdots$, let

$$A_{k,j} = \{r_{kj} = r_{kj+1} = \cdots = r_{kj+k-1} = 1\}.$$

Since, for each k, $\{A_{k,j}\}_{j=0}^{\infty}$ is independent, and

$$\sum_{j=0}^{\infty} m(A_{k,j}) = \sum_{j=0}^{\infty} 2^{-k} = \infty,$$

the Borel-Cantelli Lemma implies that there exists J_k such that

$$m\left(\bigcup_{j=0}^{J_k - 1} A_{k,j}\right) \geqq \frac{1}{2}.$$

For $k=1, 2, \cdots$, define

$$g_k(x) = 2^{kJ_k} \quad \text{if } x \in (0, 2^{-k-kJ_k}),$$
$$= 0 \quad \text{otherwise.}$$

Thus we have

$$m\left\{\sup_{m,n} E(g_k \mid r_n, \cdots, r_{n+m}) > \frac{1}{2}\right\} \geqq m\left(\bigcup_{j=0}^{J_k-1} A_{k,j}\right) \geqq \frac{1}{2},$$

but

$$\int_0^1 |g_k|\, dx = 2^{-k} \to 0 \quad \text{as } k \to \infty.$$

This shows that $f \to \sup_{n,m} E(|f| \mid r_n, \cdots, r_{n+m})$ is not of weak type (1, 1).

If we relaxed the block condition on the permutations $\{\sigma_N\}$, $f \to f^*$ would not be of weak type (p, p) for any $p \geqq 1$. We consider the operator

$$f \to \sup_{0 \leqq j < m; m} E(|f| \mid r_0, \cdots, r_{j-1}, r_{j+1}, \cdots, r_m).$$

Let

$$g_n(x) = 1 \quad \text{if } x \in (0, 2^{-n-1}),$$
$$= 0 \quad \text{otherwise.}$$

Then

$$\sup_{0 \leqq j < n} E(g_n \mid r_0, \cdots, r_{j-1}, r_{j+1}, \cdots, r_n)(x)$$
$$= \frac{1}{2} \quad \text{if } x \in (0, 2^{-n-1}) \cup \bigcup_{j=1}^{n} (2^{-j}, 2^{-j} + 2^{-n-1}),$$
$$= 0 \quad \text{otherwise.}$$

Therefore,

$$m\left\{\sup_{0 \leqq j < m} E(g_n \mid r_0, \cdots, r_{j-1}, r_{j+1}, \cdots, r_m) > \frac{1}{4}\right\} \geqq (n+1)2^{-n-1}.$$

However, $\int_0^1 |g_n|^p\, dx = 2^{-n-1}$. This verifies our statement.

To prove Theorem 1, it is sufficient to have the L^p boundedness $(p \geqq 2)$ of a weaker operator

$$f \to f^{**} = \sup_{0 \leqq m < N; N} E(|f_N| \mid r_{\sigma_N(0)}, \cdots, r_{\sigma_N(m)}),$$

where $f_N = E(f \mid r_0, \cdots, r_N) - E(f \mid r_0, \cdots, r_{N-1})$. Note that $f^{**} \leqq f^*$.

LEMMA 3. *For* $2 \leqq p \leqq \infty$,

$$\|f^{**}\|_p \leqq 2\, \|f\|_p, \qquad f \in L^p.$$

PROOF. For $p=2$,

$$\int_0^1 |f^{**}|^2\,dx \leqq \sum_{N=1}^{\infty} \int_0^1 \sup_{0\leqq m<N-1} |E(|f_N| \mid r_{\sigma_N(0)}, \cdots, r_{\sigma_N(m)})|^2\,dx$$
$$\leqq 4\sum_{N=1}^{\infty} \int_0^1 |f_N|^2\,dx = 4\int_0^1 |f|^2\,dx,$$

by Doob's inequality [**10**, p. 91]. For $p=\infty$,

$$\|f^{**}\|_\infty \leqq \|f^*\|_\infty \leqq \|f\|_\infty.$$

These norm inequalities together with the Riesz convexity theorem [**2**] imply our lemma.

REFERENCES

1. L. A. Balašov, *On series with respect to a Walsh system with monotone coefficients*, Sibirsk. Mat. Ž. **12** (1971), 25–39=Siberian Math. J. **12** (1971), 18–28. MR **44** #1982.

2. A. P. Calderón and A. Zygmund, *A note on the interpolation of sublinear operators*, Amer. J. Math. **78** (1956), 282–288. MR **18**, 586.

3. L. Carleson, *On the convergence and growth of partial sums of Fourier series*, Acta Math. **116** (1966), 135–157. MR **33** #7774.

4. K. L. Chung, *A course in probability theory*, Harcourt, Brace and World, New York, 1968. MR **37** #4842.

5. J. Gosselin and W. S. Young, *On rearrangements of Vilenkin-Fourier series which preserve almost everywhere convergence* (to appear).

6. R. A. Hunt, *On the convergence of Fourier series*, Orthogonal Exansions and their Continuous Analogues (Proc. Conf., Edwardsville, Ill., 1967), Southern Illinois Univ. Press, Carbondale, Ill., 1968, pp. 235–255. MR **38** #6296.

7. ———, *Almost everywhere convergence of Walsh-Fourier series of L^2 functions*, Actes, Congrès Intern. Math. 1970 (2), 655–661.

8. K. H. Moon, *Maximal functions related to certain linear operators*, Doctoral Dissertation, Purdue University, West Lafayette, Ind., 1972.

9. P. Sjölin, *An inequality of Paley and convergence a.e. of Walsh-Fourier series*, Ark. Mat. **7** (1969), 551–570. MR **39** #3222.

10. E. M. Stein, *Topics in harmonic analysis related to the Littlewood-Paley theory*, Ann. of Math. Studies, no. 63, Princeton Univ. Press, Princeton, N.J.; Univ. of Tokyo Press, Tokyo, 1970. MR **40** #6176.

11. W. S. Young, *Maximal inequalities and almost everywhere convergence*, Doctoral Dissertation, Purdue University, West Lafayette, Ind., 1973.

12. ———, *On the a.e. convergence of Walsh-Kaczmarz-Fourier series*, Proc. Amer. Math. Soc. (to appear).

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