ON THE PARTIAL COMPACTIFICATION OF THE ARITHMETIC QUOTIENT OF A PERIOD MATRIX DOMAIN[1]

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In his survey paper [4], P. A. Griffiths conjectured (§9.2) the existence of a partial compactification for the arithmetic quotients of period matrix domains. In this paper we want to announce some results concerning the topological aspects of this conjecture for the case of the periods of 2-forms on a polarized Hodge manifold V.

The period matrix domain D of all possible period matrices for the primitive harmonic 2-forms on V can be described as follows (cf. [3], [6]). Let $h=h^{2,0}$, $k=h^{1,1}$, where $h^{p,q}$ denotes the dimension of the space of primitive harmonic forms of bidegree (p, q). Let H be a complex vector space of dimension $m=2h+k$, $H_R \subset H$ a real form, and $H_Z \subset H_R$ a lattice. We also fix a rationally defined, nondegenerate, bilinear symmetric form Q on H. We denote by X the set of all $W \in \mathrm{Gr}(h, H)$ satisfying

$$\text{(i)} \qquad Q(W, W) = 0.$$

The algebraic group $G_C=$orthogonal group of Q, acts transitively on X, and therefore X is a projective manifold (in fact X is a Kähler C-space, cf. [5]). The period matrix domain D is the open subset of X consisting of all those points which satisfy

$$\text{(ii)} \qquad Q(W, \overline{W}) > 0.$$

(i) and (ii) are called the Riemann bilinear relations.

The noncompact real form of G_C given by: $G = \{T \in G \mid TH_R \subset H_R\}$ acts transitively on D (cf. [3]). Thus D is an open, homogeneous, complex manifold of the form

$$D = G/H = SO(2h, k)/U(h) \times SO(k).$$

Let $K \subset G$ be the unique (cf. [5]) maximal compact subgroup of G containing H. Then

$$R = G/K = SO(2h, k)/SO(h) \times SO(k)$$

is a Riemannian symmetric space of the noncompact type, dual to the real Grassmannian manifold $SO(m)/SO(2h) \times SO(k)$ (cf. [7]). Moreover R can be given explicitly as

$$R = \{U \in \mathrm{Gr}(2h, H_R) \mid Q(U, U) > 0\}.$$

The natural projection $\pi: D \to R$ given by the inclusion $H \subset K$ can be described as: $\pi(W) = U$ if and only if $W \oplus \overline{W} = U + iU$; $W \in D$, $U \in R$.

Let $\bar{D}$ (resp. $\bar{R}$) denote the topological closure of D (resp. R) in X (resp. $\mathrm{Gr}(2h, H_R)$). Let $\partial D = \bar{D} - D$, $\partial R = \bar{R} - R$.

If $W_0 \subset H$ is a totally isotropic subspace (i.e. $Q(W_0, W_0) = 0$; $Q(W_0, \overline{W}_0) = 0$), then we denote $F(W_0)$ by

$$F(W_0) = \{W \in \partial D \mid W = W_0 \oplus W_1; \ Q(W_1, \overline{W}_1) > 0\}.$$

If $U_0 \subset H_R$ is an even dimensional, totally isotropic subspace, then we denote $F_D(U_0)$ by $\bigcup F(W_0)$, where the union runs through all totally isotropic subspaces $W_0 \subset H$ such that $W_0 \oplus \overline{W}_0 = U_0 + iU_0$. We call $F_D(U_0)$ the preboundary component of D associated to U_0. We will also denote $\mathscr{F}_R(U_0)$ by

$$\mathscr{F}_R(U_0) = \{U \in \bar{R} \mid U = U_0 \oplus U_1; \ Q(U_1, U_1) > 0\}.$$

$\mathscr{F}_R(U_0)$ is called the boundary component of R associated to U_0. We define an equivalence relation in $F_D(U_0)$ by: $W \sim W'$ if and only if

$$W + U_0 + iU_0 = W' + U_0 + iU_0; \quad W, W' \in F_D(U_0).$$

$\mathscr{F}_D(U_0) = F_D(U_0)/\sim$ is called the boundary component of D associated to U_0.

Finally we will denote

$$N(\mathscr{F}_D(U_0)) = \{T \in G \mid T(\mathscr{F}_D(U_0)) = \mathscr{F}_D(U_0)\},$$
$$Z(\mathscr{F}_D(U_0)) = \{T \in G \mid T[W] = [W], \forall [W] \in \mathscr{F}_D(U_0)\},$$
$$G(\mathscr{F}_D(U_0)) = N(\mathscr{F}_D(U_0))/Z(\mathscr{F}_D(U_0)).$$

THEOREM 1. *Let U_0 be a 2ν-dimensional totally isotropic subspace of H_R. Let $\mathscr{F} = \mathscr{F}_D(U_0)$, then:*

(i) *$N(\mathscr{F})$ is a parabolic subgroup of G acting transitively on $\mathscr{F}$.*

(ii) *$G(\mathscr{F}) \cong SO(2(h-\nu), k-2\nu)$ and therefore is a real semisimple Lie group.*

(iii) *$\mathscr{F} = G(\mathscr{F})/H(\mathscr{F})$ where $H(\mathscr{F})$ is a compact subgroup of $G(\mathscr{F})$ isomorphic to $U(h-\nu) \times SO(k-2\nu)$.*

DEFINITION. Let $\Gamma \subset G$ be the arithmetic subgroup of G defined by $\Gamma = \{T \in G \,|\, TH_Z \subset H_Z\}$. We say that a boundary component $\mathcal{F}_D(U_0)$ or $\mathcal{F}_R(U_0)$ is Γ-rational if the subspace $U_0 \subset H_R$ is rational. We will denote by D^* (resp. R^*) the union of D (resp. R) and all its Γ-rational boundary components. One can prove (cf. [1], [2]) that if $\mathcal{F}_D(U_0)$ is a Γ-rational boundary component then

$$\Gamma(\mathcal{F}_D(U_0)) = \Gamma \cap N(\mathcal{F}_D(U_0))/\Gamma \cap Z(\mathcal{F}_D(U_0))$$

is an arithmetic subgroup of $G(\mathcal{F}_D(U_0))$.

Generalizing similar results known for symmetric spaces (cf. [1], [8]) we can prove the following:

THEOREM 2. *There exists an open Γ-fundamental domain $\Omega \subset D$ such that:*

(i) *$\bar{\Omega}$ intersects only rational preboundary components of D and just a finite number of them.*

(ii) *If $F_D(U_0)$ is a Γ-rational preboundary component of D such that $\bar{\Omega} \cap F_D(U_0) \neq \varnothing$, then $\Omega(\mathcal{F}_D(U_0)) = \bar{\Omega} \cap F_D(U_0)/\sim$ is a $\Gamma(\mathcal{F}_D(U\text{-}))_0$ fundamental domain in $\mathcal{F}_D(U_0)$.*

Using Theorem 2 we can then define a Satake topology on $\Gamma \backslash D^*$ as in [1] and [8], and we can prove the following:

THEOREM 3. *The quotient $\Gamma \backslash D^*$ endowed with the Satake topology has the following properties:*

(i) *$\Gamma \backslash D^*$ is locally compact and Hausdorff.*

(ii) *$\Gamma \backslash D \subset \Gamma \backslash D^*$ is an open, everywhere dense subset.*

(iii) *$\Gamma \backslash D^*$ is the finite union of subspaces of the form $\Gamma(\mathcal{F}_i) \backslash \mathcal{F}_i$ where the $\mathcal{F}_i$'s run through a set of representatives of Γ-equivalence classes of Γ-rational boundary components of D.*

(iv) *The map $\pi: D \to R$ extends to a continuous map $\pi^*: \Gamma/D^* \to \Gamma \backslash R^*$ where $\Gamma \backslash R^*$ is endowed with the Satake topology.*

Details and proofs will appear in a later publication.

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