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FUNCTIONS ON MULTIPLIERS OF p-FOURIER ALGEBRAS

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I. **Introduction.** Let G denote a locally compact abelian (LCA) group and let Γ denote the group which is dual to G. If $1 \leqq p \leqq \infty$, let $L_p(G)$ (or $L_p(\Gamma)$) denote the space of p-power integrable functions with respect to Haar measure on G (or on Γ); let $C(G)$ denote the algebra of bounded continuous functions on G and let $C_0(G)$ consist of those functions in $C(G)$ which vanish at infinity. In [2] and [3] Figa-Talamanca and Figa-Talamanca and Gaudry studied the p-Fourier algebra

$$A_p(\Gamma) = [L_p(\Gamma) \hat{\otimes} L_{p'}(\Gamma)]/K$$

where $1 \leqq p < \infty, p'$ is conjugate to p, $\hat{\otimes}$ is the projective tensor product, and K denotes the kernel of the convolution operator $c: L_p \otimes L_{p'} \to C_0(\Gamma)$ by $c(f \otimes g)(y) = f * g(y)$. $A_p(\Gamma)$ carries the quotient norm. Herz [7] showed that $A_p(\Gamma)$ is a Banach algebra under pointwise multiplication. In [2] Figa-Talamanca showed that the dual space of $A_p(\Gamma)$ is isometrically isomorphic to the space $M_p(\Gamma)$ of bounded, translation invariant, linear operators on $L_p(\Gamma)$ and that the weak operator topology on $M_p(\Gamma)$ and the weak*-topology agree on bounded sets. Implicit in [2] is the fact that $A_2(\Gamma)$ is isometrically isomorphic with $A(\Gamma)$, the algebra of Fourier transforms of integrable functions on G; $A(\Gamma)$ is equipped with the inherited norm; see also [1]. Hewitt's factorization theorem is used in [3] to prove that $A_1(\Gamma)$ is $C_0(\Gamma)$ ($C(\Gamma)$ when Γ is compact).

Let $B_p(\Gamma)$ denote the algebra of functions f in $C(\Gamma)$ which satisfy: $f(\gamma)h(\gamma) \in A_p(\Gamma)$ whenever $h \in A_p(\Gamma)$. $B_p(\Gamma)$ is a commutative and semi-simple Banach algebra under pointwise addition and multiplication when it is equipped with the operator norm; $B_p(\Gamma)$ is the algebra of bounded multipliers on $A_p(\Gamma)$. If $1 < p < \infty$ and if p' denotes the index conjugate to p, then $A_{p'} = A_p$ and $B_{p'} = B_p$, so that we may restrict p to $1 < p < 2$. It is easy to see that $B_1(\Gamma) = C(\Gamma)$, and Helson's theorem [11, p. 73] says that $B_2(\Gamma)$ is the algebra of Fourier transforms of bounded measures on G with the inherited norm. Since the inclusions $A_2(\Gamma) \subset A_p(\Gamma) \subset A_1(\Gamma)$ are continuous if $1 < p < 2$, it follows that the maximal

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ideal space of $A_p(\Gamma)$ can be identified with Γ. Thus $B_p(\Gamma)$ is an algebra of multipliers in the sense of Larsen [**9**]; we refer to this monograph for the basic facts regarding multiplier algebras.

The purpose of this note is to show that the only complex valued functions of a complex variable which operate on (the Gel'fand transforms of) $B_p(\Gamma)$ are entire functions when Γ is not compact. If Γ is compact, $A_p(\Gamma) = B_p(\Gamma)$ and the class of functions which operates on $B_p(\Gamma)$ is less restrictive. When Γ is not compact, this will imply that the algebra $B_p(\Gamma)$ is not selfadjoint and not regular and that Γ is not dense in the maximal ideal space of $B_p(\Gamma)$. The basic result (Theorem 1, below) from which this information follows can also be used to describe the isometric isomorphisms of $A_p(\Gamma)$ onto $A_p(\Lambda)$ when Λ is a second LCA group. A detailed development of these topics will be given in [**5**].

II. **Functions operating on multipliers.** In [**6**] Hahn proved that if $1 \leqq p \leqq 2$ and if $f \in L_p(\Gamma)$ and $g \in L_{p'}(\Gamma)$, then $h(y) = f * g(y)$ is the Fourier transform of an operator T_h in $M_r(G)$ if $|1/r - \frac{1}{2}| \leqq |1/p - 1|$ for which $\|T_h\|_r \leqq \|f\|_p \|g\|_{p'}$. We have the following extension of this result.

THEOREM 1. *Let* $1 \leqq p \leqq 2$ *and let* $|1/r - \frac{1}{2}| \leqq |1/p - 1|$. *Then* f *in* $B_p(\Gamma)$ *is the Fourier transform of an operator* T_f *in* $M_r(G)$ *and the map* $f \to T_f$ *faithful, norm decreasing representation of* $B_p(\Gamma)$ *as an algebra of bounded, translation invariant operators on* $L_r(G)$.

It is clear that Hahn's map $h \to T_h$ extends to a norm decreasing isomorphism of $A_p(\Gamma)$ into $M_r(G)$. Now Hahn's Lemma 1 and the fact that $A_p(\Gamma)$ has a bounded approximate identity (since $A_2(\Gamma)$ does) can be used to extend $h \to T_h$ to all of $B_p(\Gamma)$.

The Fourier transforms of bounded measures on G are elements of $B_p(\Gamma)$; denote this subalgebra by $B_2(\Gamma)$.

THEOREM 2. *Suppose that* Γ *is not compact and that* $1 < p < 2$. *Let* F *be a complex valued function defined on* $[-1, 1]$ *for which* $F(\hat{\mu}(\gamma)) \in B_p(\Gamma)$ *for every* $\hat{\mu} \in B_2(\Gamma)$ *with range in* $[-1, 1]$. *Then* F *admits an extension to all of* $\mathbf{C}$ *as an entire function.*

This follows from Igari's Theorem 1 of [**8**] and from Theorem 1 above. When Γ is compact but not discrete, F must be analytic in a neighborhood of $[-1, 1]$.

Now, by reasoning as in [**11**, Chapter 6] or as in [**8**], one may conclude that

THEOREM 3. *Suppose that* Γ *is noncompact and that* $1 < p < 2$. *Then*:

1. *If* F *is a complex valued function of a complex variable for which* $F(\hat{f})$ *is the Gel'fand transform of a function in* $B_p(\Gamma)$ *whenever* $\hat{f}$ *is, then* F *is an entire function.*

2. *For any complex number z there is a real valued function f in $B_p(\Gamma)$ and a complex homomorphism h of $B_p(\Gamma)$ for which $h(f) = z$.*

3. *$B_p(\Gamma)$ is not selfadjoint and not regular.*

4. *Γ is not dense in the maximal ideal space of $B_p(\Gamma)$.*

5. *There is a function f in $B_2(\Gamma) \subset B_p(\Gamma)$ with $f \geqq 1$ on Γ for which f^{-1} is not in $B_p(\Gamma)$.*

The analogy between $B_p(\Gamma)$ and $B_2(\Gamma)$ does not end here, but we shall wait to describe the situation more completely in [**5**].

III. **Isomorphism.** Let Γ and Λ be LCA groups and let $A_p(\Gamma)$ and $A_p(\Lambda)$ denote their respective p-Fourier algebras for $1 < p < 2$. It follows from Theorem 1 of §II and from a theorem of Strichartz [**12**] that the only isometric multipliers on $A_p(\Gamma)$ are complex unit multiples of characters on Γ. This is the basic fact needed to prove

THEOREM 4. *If Φ is an isometric isomorphism of $A_p(\Gamma)$ onto $A_p(\Lambda)$, then there is a topological isomorphism α of Λ onto Γ and an element γ_0 of Γ such that $\Phi(h)(\lambda) = h(\gamma_0 + \alpha(\lambda))$.*

COROLLARY 4.1. *Γ is topologically isomorphic to Λ if and only if $A_p(\Gamma)$ is isometrically isomorphic to $A_p(\Lambda)$; i.e. $A_p(\Gamma)$ determines Γ.*

The proof relies on the facts that Φ extends to $B_p(\Gamma)$, that Φ maps isometric multipliers to isometric multipliers, and that G is topologically isomorphic to the multipliers $\{g_0(\gamma) \mid g_0 \in G\}$ when this group is equipped with the strong operator topology.

$A_p(\Gamma)$, $B_p(\Gamma)$, and $M_p(\Gamma)$ forms an interrelated system of algebras which we first studied in [**4**]. There, we proved Theorem 1 of §II in the context of a continuity theorem of Lévy type for $M_r(G)$. No applications of Theorem 1 were given in [**4**].

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