BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 4, July 1973

THE TANGENT SPACE TO A C^k MANIFOLD

BY LAIRD E. TAYLOR

Communicated by S. S. Chern, December 27, 1972

The algebraic tangent space to a finite dimensional C^k manifold ($1 \leqq k \leqq \infty$) is the vector space of linear derivations on C_p^k, the ring of germs of real C^k functions at p. In this note we give a short proof that, for $k < \infty$, the algebraic tangent space is infinite dimensional.

This result is well known but we believe the proof presented here is the easiest. An apparently incorrect proof was given by Papy in [**3**]. A proof for the case $k = 1$ was given by Osborn in [**2**]. A complete solution is given by Newns and Walker in [**1**].

Let I be the maximal ideal of C_p^k. One sees (as in, for example, [**4**, p. 13]) that the algebraic tangent space is canonically isomorphic to $(I/I^2)^*$. We shall show I/I^2 is infinite dimensional. There is no loss of generality in assuming the manifold is the real line with coordinate x at p. Now if $\boldsymbol{f} \in I$, define the order of $\boldsymbol{f}$ by

$$o(\boldsymbol{f}) = \sup\left\{ \alpha : \lim_{x \to 0} f(x)|x|^{-\alpha} = 0 \right\}$$

where of course f is any representative for $\boldsymbol{f}$.

LEMMA. *If $\boldsymbol{f} \in I^2$ then $o(\boldsymbol{f}) > k + 1$ or is an integer.*

PROOF. Note that if $\boldsymbol{g} \in I$ we can, using Taylor's theorem, write $\boldsymbol{g} = \sum_1^{k-1} a_j \boldsymbol{x}^j + \boldsymbol{x}^k \boldsymbol{r}$ where a_j are real and $\boldsymbol{r}$ is the germ of a continuous function at p. But it follows that we can write $\boldsymbol{f} = \sum_2^k a_j \boldsymbol{x}^j + \boldsymbol{x}^{k+1}\boldsymbol{r}$. If all the a_j vanish $o(\boldsymbol{f}) \geqq k + 1$. If not, $o(\boldsymbol{f}) = \min\{j : a_j \neq 0\}$.

We now show I/I^2 is infinite dimensional by proving $\{|\boldsymbol{x}|^\sigma : k < \sigma < k + 1\}$ is a linearly independent collection. For, if not, we have $\sum_1^n c_i |\boldsymbol{x}|^{\sigma_i} = \boldsymbol{g}$ where no c_i vanishes, the σ_i are distinct, and $\boldsymbol{g} \in I^2$. But then $o(\boldsymbol{g}) = \min_i \sigma_i$ which is a noninteger less than $k + 1$ contradicting the lemma.

REFERENCES

1. W. F. Newns and A. Walker, *Tangent planes to a differentiable manifold*, J. London Math. Soc. **31** (1956), 400–407. MR **18**, 821.

2. H. Osborn, *Intrinsic characterizations of tangent spaces*, Proc. Amer. Math. Soc. **16** (1965), 591–594. MR **31** #3963.

3. G. Papy, *Sur la définition intrinsèque des vecteors tangents à une variété de classe C^r lorsque* $1 \leqq r < \infty$, C. R. Acad. Sci. Paris **242** (1956), 1573–1575. MR **17**, 892.

4. F. Warner, *Foundations of differentiable manifolds and Lie groups*, Scott, Foresman, and Company, Glencoe, Illinois, 1971.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH CAROLINA, CHAPEL HILL, NORTH CAROLINA 27514

AMS (MOS) subject classifications (1970). Primary 58A05; Secondary 16A72.
Key words and phrases. Tangent space, manifold of class C^k, derivation.