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ON CHARACTERISTIC CLASSES OF Γ-FOLIATIONS

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1. **Introduction.** In this note we describe and apply a general procedure for constructing characteristic classes of Γ-foliations. Our work was inspired by that of Godbillon-Vey [4] and Gelfand-Fuks [5] on the one hand, and that of Chern-Simons [3] on the other.

We know of several essentially equivalent procedures—some more in the line of [3] (see for example [2])—but the one we present here fits naturally and well nigh unavoidably into the Gelfand-Fuks theory and we have been told that Malgrange and Gelfand have also, quite independently, come upon it.

We will take the following point of view toward characteristic classes and foliations. Consider a pseudogroup Γ whose elements are diffeomorphisms of open sets in $\boldsymbol{R}^n$. A Γ-foliation on a smooth (that is C^∞) manifold M is by definition a maximal family F of submersions

$$f_U : U \to \boldsymbol{R}^n$$

of open sets U in M, such that for every $x \in U \cap V$ there exists an element $\gamma_{VU} \in \Gamma$ with $f_V = \gamma_{VU} \circ f_U$ in some vicinity of x.

Given two Γ-foliations, F on M and F' on M', a morphism from F to F' is by definition a smooth map

$$f : M \to M'$$

such that for every "local projection" $f'_U \in F'$ the composition

$$f'_U \circ f : f^{-1}U \to \boldsymbol{R}^n$$

is in F.

With this concept of morphism the Γ-foliations form a category $C(\Gamma)$ and we define a characteristic class of Γ-foliations with coefficients in a group $\boldsymbol{R}$, as a natural transformation

$$\alpha : C(\Gamma) \to H^*(\ ; \boldsymbol{R}).$$

Thus $\alpha(F) \in H^*(M; \boldsymbol{R})$ and if $f : F' \to F$ is a morphism, then

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$$\alpha(f^{-1}F) = f^*\alpha(F).$$

Alternatively, one could define this notion in terms of the classifying space $B\Gamma$—which we assume to be realized as a CW-complex [6].

PROPOSITION *There is a natural bijection of the R-valued characteristic classes on Γ-foliations and the cohomology group $H^*(B\Gamma; \boldsymbol{R})$.*

2. **Relation with the cohomology of formal vector-fields.** Suppose now that Γ is a transitive Lie-pseudogroup acting on $\boldsymbol{R}^n$ and let $\mathfrak{a}(\Gamma)$ denote "the Lie algebra of formal Γ vector-fields" associated to Γ. Here a vector-field defined on $U \subset \boldsymbol{R}^n$ is called a Γ vector-field if the local one parameter group which it engenders is in Γ, and $\mathfrak{a}(\Gamma)$ is defined as the inverse limit

$$\mathfrak{a}(\Gamma) = \varprojlim \mathfrak{a}^k(\Gamma)$$

of the k jets at 0 of Γ vector-fields. In the pseudogroup Γ let Γ_0 be the set of elements of Γ keeping 0 fixed and set Γ_0^k equal to the k jets of elements in Γ_0.

Then the Γ_0^k form an inverse system of Lie groups and we can find a subgroup $K \subset \varprojlim \Gamma_0^k$, whose projection on every Γ_0^k is a maximal compact subgroup for $k > 0$. This follows from the fact that the kernel of the projection $\Gamma_0^{k+1} \to \Gamma_0^k$ is a vector space for $k > 0$. The subgroup K is unique up to conjugation, and its Lie algebra k can be identified with a subalgebra of $\mathfrak{a}(\Gamma)$.

For our purposes we now need the cohomology of basic elements rel K in $\mathfrak{a}(\Gamma)$, $H(\mathfrak{a}(\Gamma); K)$, which is defined in the following manner: Let $A\{\mathfrak{a}^k(\Gamma)\}$ denote the algebra of multilinear alternate forms on $\mathfrak{a}^k(\Gamma)$, and let $A\{\mathfrak{a}(\Gamma)\}$ be the direct limit of the $A\{\mathfrak{a}^k(\Gamma)\}$. The bracket in $\mathfrak{a}(\Gamma)$ there induces a differential on $A\{\mathfrak{a}(\Gamma)\}$ in the usual way, and we write $H\{\mathfrak{a}(\Gamma)\}$ for the resulting cohomology group. The relative group $H^*(\mathfrak{a}(\Gamma); K)$ is now defined as the cohomology of the subcomplex of $A\{\mathfrak{a}(\Gamma)\}$ consisting of elements which are invariant under the natural action of K, and annihilated by all inner products with elements of k.

THEOREM 1. *Let F be a Γ-foliation on M. There is an algebra homomorphism*

$$\varphi: H\{\mathfrak{a}(\Gamma); K\} \to H(M; \boldsymbol{R})$$

which is a natural transformation on the category $C(\Gamma)$.

Note that $H(\mathfrak{a}(\Gamma); K)$ can be thought of as the continuous cohomology of the topological category Γ.

To construct φ one proceeds as follows:

Let $P^k(\Gamma)$ be the differentiable bundle of k jets at the origin of elements of Γ. It is a principal Γ_0^k-bundle. On the other hand, Γ acts transitively on the left on $P^k(\Gamma)$.

Denote by $A(P^\infty(\Gamma))$ the direct limit of the algebras $A(P^k(\Gamma))$ of differential forms on $P^k(\Gamma)$. The invariant forms with respect to the action of Γ constitute a differential subalgebra denoted by A_Γ.

Proposition. *A_Γ is naturally isomorphic to $A(\mathfrak{a}(\Gamma))$.*

Now let F be a foliation on M. Let $P^k(F)$ be the differentiable bundle over M whose fiber over $x \in M$ is the space of k jets at x of local projections $f_U \in F$ such that $f_U(x) = 0$. This is a Γ_0^k-principal bundle. Its restriction to U is isomorphic to the inverse image by f_U of the bundle $P^k(\Gamma)$: If g^k is the k jet at 0 of an element g of Γ and $g(0) \in f_U(U)$, the isomorphism maps (x, g^k) on the k jet at x of $g^{-1}f_U$. Hence it is readily seen that the differential algebra of Γ-invariant forms on $P^k(\Gamma)$ is mapped in the algebra $A(P^k(F))$ of differential forms on $P^k(F)$.

Hence if we denote by $A(P^\infty(F))$ the direct limit of the $A(P^k(F))$, we get an injective homomorphism φ of $A(\mathfrak{a}(\Gamma))$ in $A(P^\infty(F))$ commuting with the differential.

This homomorphism is compatible with the action of K, hence induces a homomorphism on the subalgebra of K-basic elements. But the algebra $A(P^k(F); K)$ of K-basic elements in $A(P^k(F))$ is isomorphic to the algebra of differential forms on $P^k(F)/K$ which is a bundle over M with contractible fiber Γ_0^k/K. Hence $H(A(P^k(F); K))$ is isomorphic, via the de Rham theorem, to $H(M; \boldsymbol{R})$.

The homomorphism φ is therefore obtained as the composition

$$H(\mathfrak{a}(\Gamma); K) \to H(A(P^\infty(F); K)) = H(M; \boldsymbol{R}).$$

Remark. In the same spirit, one can for instance also consider the category of differentiable fiber bundles E, with fiber a compact comogeneous space $B = G/K$, with structural group G, and a foliation transverse to the fibers.

Also let v_B be the topological Lie algebra of tangent vector fields on B (cf. [5]). Then, there is a natural homomorphism

$$H(v_B; K) \to H(E)$$

which is naturally related to the preceding one.

3. **The basic examples.** By extending the computation of Gelfand-Fuks [5] the group $H(\mathfrak{a}; K)$ can be computed in certain cases, and we will describe the results in several instances of topological interest; namely when Γ consists of the holomorphic, orientation preserving, and arbitrary diffeomorphisms of open sets in $\boldsymbol{R}^n$. We denote these Γ's by $\Gamma_n\boldsymbol{C}, \Gamma_n^+$ and Γ_n respectively.

In the following $E(u_1, u_2, \ldots)$ denotes an exterior algebra in the indicated u's and $\boldsymbol{R}[c_1, \ldots, c_n]$ denotes the polynomial ring over $\boldsymbol{R}$ in the variables $c_1, \ldots, c_q$, with degree $c_i = 2i$. We also write $\hat{\boldsymbol{R}}[c_1, \ldots, c_n]$ for the ring $\boldsymbol{R}[c_1, \ldots, c_n]$/(elements of degree $> 2q$), and denote by WO_n the differential complex:

$$WO_n = E(u_1, u_3, \ldots, u_{2k+1}) \otimes \hat{\boldsymbol{R}}[c_1, \ldots, c_n],$$

$dc_i = 0$; $du_i = c_i$ for odd i; $2k + 1$ largest odd integer $<n$. With this understood, one then has the

THEOREM 2. *The K's for Γ_n and Γ_n^+ are the orthogonal and special orthogonal groups O_n and SO_n respectively, and*

$$H^*(\mathfrak{a}(\Gamma_n); K) = H(WO_n),$$

while

$$H^*(\mathfrak{a}(\Gamma_n^+); K) = H(WO_n), \qquad n \text{ odd},$$

$$= H(WO_n)[\chi]/(\chi^2 - c_n), \qquad n \text{ even}.$$

REMARKS. (1) Every n-manifold of course carries a natural Γ_n structure, the one whose projections contain all the coordinate charts of M. Under φ the c_i's above then go into the Chern classes of the complexified tangent bundle to M, while χ goes over to the Euler class.

For a general Γ-foliation F on M, the c_i's and χ play the corresponding role for the normal bundle νF to F. The truncation of $\boldsymbol{R}[c_1, \ldots, c_n]$ thus expresses and refines the vanishing theorem of [5]. Finally the invariant of Godbillon-Vey is identified with the image of the class $\mu_1 c_1^{2n-1}$ under φ.

(2) The Gelfand-Fuks computations for $H\{\mathfrak{a}(\Gamma_n)\}$ can also be expressed conveniently in terms of a complex. Indeed, define W_n by

$$W_n = E(u_1, u_2, \ldots, u_n) \otimes \hat{\boldsymbol{R}}[c_1, \ldots, c_n], \qquad du_i = c_i.$$

Then $H\{\mathfrak{a}(\Gamma_n)\} \cong H(W_n)$. This group occurs naturally in our context when one seeks characteristic classes for Γ-foliations with trivialized normal bundles.

We turn next to the complex case. Here one has

THEOREM 3. *For $\Gamma_n \boldsymbol{C}$ the unitary group plays the role of K, and*

$$H^*(\Gamma_n \boldsymbol{C}; K) \otimes \boldsymbol{C} \cong H(WU_n)$$

where WU_n is the complex $E[u_1, \ldots, u_n] \otimes \hat{\boldsymbol{R}}[c_1, \ldots, c_n] \otimes \hat{\boldsymbol{R}}[\bar{c}_1, \ldots, \bar{c}_n] \otimes \boldsymbol{C}$ with $du_k = c_k - \bar{c}_k$.

Here, of course, φ again assigns to c_i the ith Chern class of νF, while the classes of type $u_j c_1^{\alpha_1}, \ldots, c_n^{\alpha_n}$ correspond to the relative classes in the Chern-Simons theory.

4. **On the nontriviality of φ.** It seems to be very difficult, especially in

the real case, to find foliations which distinguish between the different classes defined by φ. One procedure is to extend the example of Godbillon-Vey-Roussarie, and to seek Γ-foliations on homogeneous spaces, which have nontrivial invariants after division by a suitable discrete group. From our point of view, this procedure takes the following form. Let G be a Lie group and $H \subset G$ a closed subgroup. The action of G on a vicinity of H in G/H then defines a Lie pseudogroup $\Gamma_{G,H}$ on $\boldsymbol{R}^n$, $n = \dim G/H$. For this Lie pseudogroup one, of course, finds that $\mathfrak{a}(\Gamma_{G,H}) = \mathfrak{g}$ the Lie algebra of G, while the role of K is played by the *maximal compact subgroup of H*. Consider now the inclusion $i: \Gamma_{G,H} \subset \Gamma_n$. Then our φ construction can be thought of as inducing a commutative diagram:

$$\begin{array}{ccc} H(B\Gamma_{G/H}) & \xleftarrow{i^*} & H(B\Gamma_n) \\ \uparrow \varphi & & \uparrow \varphi \\ H(\mathfrak{g}; K) & \xleftarrow{i^*} & H(\mathfrak{a}(\Gamma_n); O_n). \end{array}$$

Now suppose G is semisimple. Then according to the deep theory of Borel-Harish-Chandra, G contains a group D acting discretely on G/K and such that $D\backslash G/K$ is compact. From this fact and Poincaré duality, it follows that $H^*(\mathfrak{a}; K)$ injects into $H^*(D\backslash G/K)$ and this in turn easily leads to the conclusion:

PROPOSITION. *If G is semisimple, then*

$$\varphi: H(\mathfrak{g}; K) \to H(B\Gamma_{G/H})$$

is injective.

Using this principle, one obtains a weak independence result for the classes u^1c^α, $c^\alpha = c_1^{\alpha_1} \cdots c_n^{\alpha_n}$, $\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n$ in $H^{2n+1}(WO_n)$. For this purpose define a minimal variety V to be the quotient of a simple semisimple complex Lie group by a maximal parabolic subgroup. (Examples are the complex Grassmannians, the quadrics and, in general, each simple group G has no more than l minimal varieties associated to it where $l = \operatorname{rank} G$.) These varieties are compact so that if V has complex dimension n, then $c^\alpha(V)$ is well-defined.

INDEPENDENCE THEOREM. *The φ image of the classes $\mu_1 c^\alpha$ in $H^{2n+1}(WO_n)$ are as independent as the Chern numbers $c^\alpha(V)$ of the minimal varieties V of* $\dim_C V = n$.

COROLLARY. *In degree* $(2n + 1)$ *the image of φ has* $\dim \geqq 2$, *for* $n \geqq 2$.

These two classes correspond to the complex projective space and the quadric, which exist in every dimension. Unfortunately, they are also the only ones when n is a prime. However, these classes in turn imply that many higher dimensional ones are also nontrivial.

In the complex case the corresponding classes are much easier to detect, and there they, in fact, take on a continuum of values. Furthermore, they play a universal role for the infinitesimal Lefschetz theorem (see [**2**]). Precisely, one has the following:

THEOREM 4. *Let F_λ be the $\Gamma_n C$-structure defined on $C_{n+1} - 0$, by the vector-field*

$$X_\lambda = \lambda_0 z_0 \frac{\partial}{\partial z_0} + \lambda_2 z_2 \frac{\partial}{\partial z_2} + \cdots + \lambda_n z_n \frac{\partial}{\partial z_n} \qquad (\lambda_i \neq 0).$$

Then one has the formula

$$\int_{S^{2n+1}} \varphi(\mu_1 c^\alpha)[F_\lambda] = \Im c^\alpha(\lambda)/c^n(\lambda)$$

where $\Im$ denotes imaginary part and the $c_i(\lambda)$ are taken to be the ith elementary symmetric functions of the system $\lambda = \{\lambda_0, \ldots, \lambda_n\}$.

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