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REPRESENTATION OF H^p-FUNCTIONS

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ABSTRACT. Let E be a set of positive measure on the unit circle. Let $f \in H^p$ $(1 \leqq p \leqq \infty)$ and g be the restriction of f to E. It is shown that functions g_λ, $\lambda > 0$, can be constructed from g so that $g_\lambda \to f$. We also characterize those functions g on E which are restrictions of functions in H^p $(1 < p \leqq \infty)$.

In the following, the space H^p $(1 \leqq p \leqq \infty)$ will, according to the context, be either the Hardy class of analytic functions in the open unit disc D or the space of the corresponding boundary value functions, viz the subspace of "analytic" functions in $L^p(C)$, C being the unit circle. If $E \subset C$ has positive measure then it is well known (see [3]) that a function in H^p cannot vanish on E without being identically zero. Thus, theoretically at least, $f \in H^p$ is uniquely "determined" by its values on E. In the present work we address ourselves to the problem of recovering functions in H^p from their restrictions to E. Theorem I gives an explicit constructive solution to this problem. The allied problem of characterizing the restrictions to E of functions in H^p $(1 < p \leqq \infty)$ is solved in Theorem II. To the best of our knowledge, the only known results relating to these problems are due to the author [4] where the case $p = 2$ is dealt with.

THEOREM I. *Let $E \subset C$ with $m(E) > 0$. Suppose that $1 \leqq p \leqq \infty$, $f \in H^p$ and that g is the restriction of f to E. For each $\lambda > 0$ define analytic functions h_λ, g_λ on D by*

$$h_\lambda(z) = \exp\left\{-\frac{1}{4\pi}\log(1+\lambda)\int_E \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta\right\}, \qquad z \in D,$$

$$g_\lambda(z) = \lambda h_\lambda(z)\frac{1}{2\pi i}\int_E \frac{\bar{h}_\lambda(w)g(w)\,dw}{w-z}, \qquad z \in D.$$

Then as $\lambda \to \infty$, $g_\lambda \to f$ uniformly on compact subsets of D. Moreover for $1 < p < \infty$ we also have $\|g_\lambda - f\|_p \to 0$ as $\lambda \to \infty$.

THEOREM II. *Let $E \subset C$ with $0 < m(E) < m(C)$. For $g \in L^1(E)$ let g_λ be as in Theorem* I. (a) *If $1 < p < \infty$ then a function $g \in L^p(E)$ is the restriction to E of some $f \in H^p$ if and only if $\sup_{\lambda > 0}\|g_\lambda\|_p < \infty$.* (b) *A function $g \in L^\infty(E)$ is the restriction to E of some $f \in H^\infty$ if and only if $\sup_{p>1}\limsup_{\lambda\to\infty}\|g_\lambda\|_p < \infty$.*

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The proof of Theorem I will be based on a series of lemmas. First we recall some elementary properties of Toeplitz operators on H^p spaces (for details in the special case $p = 2$ see [1], and for the general case $1 < p < \infty$ see [5]). Let $1 < p < \infty$. For each $\varphi \in L^\infty$, the Toeplitz operator T_φ is defined by $T_\varphi f = P(\varphi f)$, $f \in H^p$, where P is the natural projection of L^p onto H^p. We need the following facts: (i) $\|T_\varphi\| \leqq C_p\|\varphi\|_\infty$, (ii) if $\varphi, \psi \in L^\infty$ and if either $\bar{\varphi} \in H^\infty$ or $\psi \in H^\infty$, then $T_{\varphi\psi} = T_\varphi T_\psi$. This latter fact immediately yields

LEMMA 1. *If h, $1/h \in H^\infty$ and $\varphi = |h|^{-2}$, then the Toeplitz operator T_φ is invertible and $T_\varphi^{-1} = T_h T_{\bar{h}}$.*

PROOF. $T_h T_{\bar{h}} T_\varphi = T_h(T_{\bar{h}} T_{1/\bar{h}}) T_{1/h} = T_h T_{1/h} = I$, etc.

Let χ_E be the characteristic function of the set E and let for $\lambda > 0$, $\varphi_\lambda = 1 + \lambda \chi_E$. Then the function h_λ defined in Theorem I satisfies, $1/\varphi_\lambda = h_\lambda \bar{h}_\lambda$. Also h_λ, $1/h_\lambda \in H^\infty$. Thus by Lemma 1, we have

LEMMA 2. *T_{φ_λ} is invertible and $T_{\varphi_\lambda}^{-1} = T_{h_\lambda} T_{\bar{h}_\lambda}$.*

LEMMA 3. *Define for each $a \in D$, $e_a(z) = 1/(1 - \bar{a}z)$, $z \in D$. Then $e_a \in H^p$, $1 \leqq p \leqq \infty$, and if T_{φ_λ} is treated as an operator on H^p $(1 < p < \infty)$, we have $T_{\varphi_\lambda}^{-1} e_a = h_\lambda(a) h_\lambda e_a$.*

PROOF. For each $g \in H^q$ $(q = p/(p-1))$, we have $(T_{\bar{h}_\lambda} e_a, g) = (e_a, h_\lambda g)$ $= h_\lambda(a)\bar{g}(a) = h_\lambda(a)(e_a, g)$. Thus $T_{\bar{h}_\lambda} e_a = h_\lambda(a) e_a$. An appeal to Lemma 2 finishes the proof.

LEMMA 4. *Let K be a compact subset of D and $1 \leqq p \leqq \infty$. Then as $\lambda \to \infty$, $\|h_\lambda(a) h_\lambda e_a\|_p \to 0$ uniformly for $a \in K$.*

PROOF. We note that $\|h_\lambda\|_\infty \leqq 1$ and $|h_\lambda(a)| \leqq (1 + \lambda)^{-\alpha}$ where $\alpha > 0$ and α depends on $|a|$.

Let now S be the Toeplitz operator on H^p $(1 < p < \infty)$ corresponding to the characteristic function χ_E of E. Then since $I + \lambda S = T_{\varphi_\lambda}$, $(I + \lambda S)^{-1}$ exists by Lemma 2. Also by Lemma 4, $\|(I + \lambda S)^{-1} e_a\|_p \to 0$ as $\lambda \to \infty$. By Lemma 2 and fact (i) about Toeplitz operators we also have

$$\|(I + \lambda S)^{-1}\| = \|T_{h_\lambda} T_{\bar{h}_\lambda}\| \leqq \|h_\lambda\|_\infty^2 C_p^2 \leqq C_p^2.$$

Noting that $\{e_a : a \in D\}$ is a fundamental set in H^p, we therefore obtain (cf., e.g., [3, p. 55]) that $\|(I + \lambda S)^{-1} f\|_p \to 0$ for every $f \in H^p$. Noting that for $f \in H^p$, $(I + \lambda S)^{-1} f = f - \lambda(I + \lambda S)^{-1} S f$, we get

LEMMA 5. *If $1 < p < \infty$ and $f \in H^p$, then as $\lambda \to \infty$,*

$$\|\lambda(I + \lambda S)^{-1} S f - f\|_p \to 0.$$

The proof of Theorem I (for $1 < p < \infty$) will be complete if we show that $g_\lambda = \lambda(I + \lambda S)^{-1}Sf$. But this is routine: For $z \in D$,

$$(\lambda(I + \lambda S)^{-1}Sf, e_z) = \lambda(Sf, (I + \lambda S)^{-1}e_z) = \lambda(\chi_E f, (I + \lambda S)^{-1}e_z)$$
$$= \lambda(f, (I + \lambda S)^{-1}e_z)_E = \lambda(f, \bar{h}_\lambda(z)h_\lambda e_z)_E.$$

In the above chain of equalities, the first is a consequence of the fact that $(I + \lambda S)^*$ is the operator $(I + \lambda S)$ on H^q ($q = p/(p-1)$) and the last results from Lemma 3. The notation $(\ ,\ )_E$ denotes the "inner product" over the set E. Now it can be readily checked that $\lambda(f, \bar{h}_\lambda(z)h_\lambda e_z)_E$ is the same as the defining expression for $g_\lambda(z)$.

The case $p = \infty$ is easy. If $f \in H^\infty$ then since f is also in H^2, by the preceding, $\|g_\lambda - f\|_2 \to 0$ and hence $g_\lambda \to f$ uniformly on compact subsets of D.

Turning to the case $p = 1$, let $f \in H^1$. For $0 < r < 1$, define f_r by $f_r(e^{i\theta}) = f(re^{i\theta})$. Then as is well known, $\|f_r\|_1 \leqq \|f\|_1$ and $\|f_r - f\|_1 \to 0$ as $r \to 1$. Let us define, for each $\lambda > 0$, $f_{r,\lambda}$ by

$$f_{r,\lambda}(z) = \lambda h_\lambda(z)\frac{1}{2\pi i}\int_E \frac{\bar{h}_\lambda(w)f_r(w)}{w - z}\,dw, \qquad z \in D.$$

Then we see that, for every compact set $K \subset D$, the following statements hold uniformly in K: (1) $f_{r,\lambda} \to g_\lambda$ as $r \to 1$, (2) $f_r \to f$ as $r \to 1$, (3) $f_{r,\lambda} \to f_r$ as $\lambda \to \infty$. The less trivial of these statements, viz. (3), follows because $f_r \in H^2$ and the case $p = 2$ of the theorem applies. If we show further that the convergence in (3) is also uniform for r in $(0, 1)$ then we can conclude that $g_\lambda \to f$ as $\lambda \to \infty$ uniformly in K and the proof of the theorem for $p = 1$ will be complete. For this purpose, remembering that $f \in H^2$ we have for each $z \in K$,

$$f_{r,\lambda}(z) - f_r(z) = (\lambda(I + \lambda S)^{-1}Sf_r - f_r, e_z) = ((I + \lambda S)^{-1}f_r, e_z)$$
$$= (f_r, (I + \lambda S)^{-1}e_z) = (f_r, \bar{h}_\lambda(z)h_\lambda e_z).$$

Hence we obtain

$$|f_r(z) - f_{r,\lambda}(z)| \leqq \|f_r\|_1\|\bar{h}_\lambda(z)h_\lambda e_z\|_\infty \leqq \|f\|_1\|\bar{h}_\lambda(z)h_\lambda e_z\|_\infty.$$

The last term is independent of r and Lemma 4 ($p = \infty$) does the job.

Proof of Theorem II. The "only if" parts are evident from Theorem I. As for the "if" part in (a), the boundedness of $\{\|g_\lambda\|_p\}$ together with the weak* compactness of closed balls in H^p provide us with a sequence $\lambda_n \to \infty$ such that g_{λ_n} converges weak* to some f in H^p. Let $g_1 \in L^p(C)$ be defined by setting $g_1 = g$ on E and $g_1 = 0$ otherwise. Denote Pg_1 by $\tilde{g}$. From the discussion following Lemma 5, it can be seen that

$$g_\lambda = \lambda(I + \lambda S)^{-1}\tilde{g}.$$

Thus for every $k \in H^q$ $(q = p/(p-1))$, $(\lambda_n(I + \lambda_n S)^{-1} S\tilde{g}, k) = (g_{\lambda_n}, Sk) \to (f, Sk) = (Sf, k)$, while by Lemma 5, the first of these inner products converges to $(\tilde{g}, k)$. Hence $\tilde{g} = Sf$. This means that the Fourier coefficients $((f - g_1)\chi_E)\hat{}(n)$ are zero for $n \geqq 0$. In other words, $(\bar{f} - \bar{g}_1)\chi_E \in H^p$. Since $m(C \backslash E) > 0$, we must have $f = g_1$ on E.

For proving the "if" part in (b) we need to make just two observations. First, $g \in L^\infty(E)$ implies $g_\lambda \in H^p$ for each $p < \infty$ and hence part (a) gives f belonging to H^p for all $p < \infty$ and such that g is the restriction to E of f. Secondly, $\|g_\lambda\|_p \to \|f\|_p$ as $\lambda \to \infty$ and $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$. The details are left to the reader.

REMARKS. 1. In the proof of Theorem I, we did not use the F. & M. Riesz Theorem. We thus obtain a new proof of the statement: if $f \in H^p$ $(1 \leqq p \leqq \infty)$, $f = 0$ on E, $m(E) > 0$, then $f = 0$.

2. Theorem I points out a way which enables us to draw conclusions about the properties of a holomorphic function from the knowledge of its values on an arc. It is possible to obtain results parallel to the classical Cauchy theory where we now have integrals over a curve which may not be closed. Details of these and other related results will be published elsewhere.

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