

AN EXISTENCE THEOREM FOR A MODIFIED
 SPACE-INHOMOGENEOUS, NONLINEAR
 BOLTZMANN EQUATION

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1. **Preliminaries.** Consider the initial value problem for the following class of nonlinear Boltzmann equations:

$$(1.1) \quad \frac{\partial}{\partial t} f + v \nabla_x f = Qf \quad (t > 0), f(0) = f_0 \geq 0,$$

with collision operator

$$(1.2) \quad Qf(x, v_1) = \int_{R^3 \times B^2} [J_u^* \langle f_1, f_2 \rangle - \langle f_1, f_2 \rangle] k \, du \, dv_2.$$

The space coordinate x belongs to a parallelepiped

$$\omega = \{x = (x_1, x_2, x_3); |x_j| \leq a_j/2\},$$

and

$$\begin{aligned} \langle f_1, f_2 \rangle &= f(x, v_1) f(x, v_2) && \text{if } |f(x, v_1) f(x, v_2)| \leq N, \\ &= N \operatorname{sign} f(x, v_1) f(x, v_2) && \text{otherwise.} \end{aligned}$$

The impact parameter u in R^2 is, for convenience, restricted to the disc

$$B^2 = \{u \in R^2; |u| \leq 1/\sqrt{\pi}\}.$$

The kernel $k(v_1, v_2)$ is a measurable, nonnegative, and bounded function vanishing for $|v_1|^2 + |v_2|^2 > K_1^2$, with $k(v_1, v_2) = k(v_2, v_1)$ and invariant under J^* , $J^*k = k$. Here J^* is induced by a C^1 -diffeomorphism J on $R^3 \times R^3 \times B^2$, restricted in a certain way. With the velocity mappings

$$\begin{aligned} 1: R^3 \times R^3 &\rightarrow R: (v_1, v_2) \rightarrow 1, \\ p: R^3 \times R^3 &\rightarrow R^3: (v_1, v_2) \rightarrow v_1 + v_2, \\ T: R^3 \times R^3 &\rightarrow R: (v_1, v_2) \rightarrow |v_1|^2 + |v_2|^2, \\ \Sigma: R^3 \times R^3 &\rightarrow R^3 \times R^3: (v_1, v_2) \rightarrow (v_2, v_1), \end{aligned}$$

the restriction on the (collision) mapping J can be written

$$(1.3) \quad 1 \circ J_u = 1, \quad p \circ J_u = p, \quad T \circ J_u = T,$$

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$$(1.4) \quad \Sigma \circ J_u = J_u \circ \Sigma,$$

$$(1.5) \quad J \circ J = \text{identity}.$$

For a discussion of the impact parameter u , the kernel k and the collision mapping J , see [1].

We now consider Q . As usual Q is zero for the Maxwell distributions, which thus are equilibrium solutions. A basic assumption in the deduction of the Boltzmann equation in kinetic gas theory is the low density of the gas. Then the contribution of the higher collisions are neglected in the development of the gas on physical grounds, so that the binary collisions alone are responsible for the development of the gas through the binary collision operator. For the same physical reasons an additional term should be inserted in case high densities develop in the gas. In the present paper that additional term cancels the binary collisions at high densities. But the proofs below hold for more general additional terms, thus under more general assumptions on the behaviour of the gas at high densities.

Boltzmann used $k = C|v_1 - v_2|$ as the weight-function in Q . In the present paper, however, we use for the physical case

$$\begin{aligned} k &= C|v_1 - v_2| && (|v_1|^2 + |v_2|^2 \leq K_1^2), \\ &= 0 && (|v_1|^2 + |v_2|^2 > K_1^2). \end{aligned}$$

We consider (1.1) written as

$$(1.6) \quad \frac{d}{dt} f(x + v_1 t, v_1, t) = Qf(x + v_1 t, v_1, t) \quad (t > 0), f(0) = f_0 \geq 0,$$

f here denoting the periodic continuation in x with period ω of the function f in (1.1). The main result is an existence and uniqueness theorem for (1.6), when $f_0 \in L^\infty(\omega \times R^3)$. We also prove the moment conservation laws and the usual laws of macroscopic gas dynamics as well as the \mathcal{H} -theorem.

As in the bounded space-homogeneous case treated in [1], the existence and uniqueness is based on the local Lipschitz continuity of Q and the positivity of a related operator. The method also works in more general situations, e.g. for a container ω with sufficiently well-behaved boundary under specular reflexion and a constant exterior force.

2. Results. We notice that $Qf(x + v_1 t, v_1, t)$ does not depend on $|v_1| > K_1$. Hence it is enough to consider initial data $f_0 \in L^\infty(\Omega)$ with

$$\Omega = \{(x, v); x \in \omega, |v| \leq K_1\},$$

and to consider the solution f for $t > 0$ as a function of v for $|v| \leq K_1$ and periodic in x with period ω .

We also notice that

$$|\langle f_1, f_2 \rangle - \langle g_1, g_2 \rangle| \leq |f(x, v_1)| |f(x, v_2) - g(x, v_2)| + |g(x, v_2)| |f(x, v_1) - g(x, v_1)|.$$

Then Q is locally Lipschitz continuous,

$$(2.1) \quad \|Qf - Qg\|_\infty \leq K_2(\|f\|_\infty + \|g\|_\infty)\|f - g\|_\infty,$$

if $\|\cdot\|_\infty$ denotes the norm in $L^\infty(\Omega)$ and $K_2 = 2 \sup k 3^{-1} 4K_1^3\pi$.

THEOREM 2.1. *There exists a unique, nonnegative solution $f(x, v, t) \in L^\infty(\Omega)$ ($t > 0$) of (1.6) for every $f_0 \geq 0$ in $L^\infty(\Omega)$.*

PROOF. The local existence follows as in the space-homogeneous case of [1]. We repeat it for the convenience of the reader. By (2.1), Q is locally Lipschitz continuous, and so there exists a unique solution f of (1.6) for $0 \leq t \leq t_0, t_0 > 0$ only depending on K_2 and $\|f_0\|_\infty$. But $\|Q(f, f)\|_\infty \leq NK_2$ and

$$(2.2) \quad \|f(t)\|_\infty \leq \|f_0\|_\infty + \int_0^t NK_2 = \|f_0\|_\infty + tNK_2,$$

$$\sup_\Omega |f(x + vt_1, v, t_1) - f(x + vt_2, v, t_2)| \leq NK_2|t_1 - t_2|.$$

We conclude by the usual continuation argument that f exists for all $t > 0$.

The proof is complete if we can show the positivity of f . Consider the equation

$$(2.3) \quad \begin{aligned} g(x + v_1t, v_1, t) = & \exp[-H(x + v_1t, t)] \cdot f_0(x, v_1) \\ & + \int_0^t \exp[-H(x + v_1t, t) + H(x + v_1s, s)] \\ & \cdot Q'g(x + v_1s, v_1, s) ds, \end{aligned}$$

with

$$(2.4) \quad H(x + vt, t) = \int_0^t \int_{|v_2| \leq K_1} f(x + v\tau, v_2, \tau) dv_2 d\tau \sup k$$

and

$$(2.5) \quad Q'g(x, v_1, s) = Qg(x, v_1, s) + \int_{|v_2| \leq K_1} g(x, v_1, s)g(x, v_2, s) dv_2 \sup k.$$

We notice that $Q'g \geq 0$, if $g \geq 0$ and that f satisfies (2.3). But by (2.1) and (2.5),

$$(2.6) \quad \begin{aligned} \|Q'f - Q'g\|_\infty &\leq \|Qf - Qg\|_\infty + K_2(\|f\|_\infty + \|g\|_\infty)\|f - g\|_\infty \\ &\leq 2K_2(\|f\|_\infty + \|g\|_\infty)\|f - g\|_\infty. \end{aligned}$$

Moreover by (2.2) and (2.4) for every $t_0 > 0$, there is a C_{t_0} , such that

$$(2.7) \quad 0 < \exp[-H(x + vt, t) + H(x + vs, s)] < C_{t_0}$$

for $0 \leq s \leq t \leq t_0$. It follows by (2.6) and (2.7) that the L^∞ -solution of (2.3) is unique. For small t it is the limit of the following convergent iteration scheme

$$\begin{aligned} g_1 &= 0, \\ g_j(x + v_1t, v_1, t) &= \exp[-H(x + v_1t, t)] \cdot f_0(x, v_1, t) \\ &\quad + \int_0^t \exp[-H(x + v_1t, t) + H(x + v_1s, s)] \\ &\quad \cdot Q'g_{j-1}(x + v_1s, v_1, s) ds, \quad j = 2, 3, \dots \end{aligned}$$

But $g_2 \geq g_1 \geq 0$ and by induction $(g_j)_1^\infty$ is a nonnegative, monotonically increasing sequence. Hence $g = \lim_{j \rightarrow \infty} g_j \geq 0$. As f is the unique solution, it follows that $g = f \geq 0$ for small t , and so by a continuation argument for all $t > 0$.

THEOREM 2.2. *The L^∞ -solution f of (1.6) satisfies for $\varphi = 1, v, |v|^2$ the moment conservation*

$$\int_\Omega f(x, v, t)\varphi(v) dx dv = \int_\Omega f_0(x, v)\varphi(v) dx dv$$

and generates a weak solution to the equation of macroscopic gas dynamics

$$\frac{\partial}{\partial t} \int f(x, v, t)\varphi(v) dv + \sum_{j=1}^3 \frac{\partial}{\partial x_j} \int f(x, v, t)v_j\varphi(v) dv = 0.$$

PROOF. Using (1.3)–(1.5) we can easily show that

$$(2.8) \quad \int Qf(x, v, t)\varphi(v) dv = 0,$$

and then using (1.6) that

$$\int_\Omega f(x + vt, v, t)\varphi(v) dv dx = \int_\Omega f_0(x, v)\varphi(v) dx dv.$$

Hence the moment conservation holds.

Multiplying (1.6) by φ and a continuously differentiable function $\psi(x + vt, t)$ with period ω in x , and then integrating and applying (2.8) we get

$$\begin{aligned} \int f(x, v, t)\varphi(v)\psi(x, t) dv dx &= \int f_0(x, v)\varphi(v)\psi(x, 0) dx dv \\ &+ \int_0^t \int f(x, v, s)\varphi(v) \frac{\partial}{\partial s} \psi(x, s) dv dx ds \\ &+ \sum_{j=1}^3 \int_0^t \int f(x, v, s)\varphi(v)v_j \frac{\partial}{\partial x_j} \psi(x, s) dv dx ds. \end{aligned}$$

These are the weak equations of macroscopic gas dynamics satisfied by f .

THEOREM 2.3. *The \mathcal{H} -function*

$$\mathcal{H}(t) = \int_{\Omega} f(x, v, t) \log f(x, v, t) dx dv$$

is nonincreasing as a function of t .

PROOF. Evidently the standard proof of this \mathcal{H} -theorem holds in a formal sense in the present case. This formal proof is a strict proof for $0 \leq t \leq t_0$, if

$$(2.9) \quad f(x, v, t) > \varepsilon > 0 \quad \text{for } (x, v) \in \Omega, 0 \leq t \leq t_0 \quad (\text{cf., e.g., [1]}).$$

By (2.2)–(2.4) the solution f_n of (1.6) with initial value $f_0 + n^{-1} e^{-v^2}$ satisfies (2.9) for any $t_0 > 0$. Hence

$$\int_{\Omega} f_n(x, v, t) \log f_n(x, v, t) dx dv$$

is nonincreasing as a function of t for $0 \leq t \leq t_0$. But $\lim_{n \rightarrow \infty} f_n = f$ in L^∞ , uniformly in t for $0 \leq t \leq t_0$. We conclude that

$$\mathcal{H}(t) = \int f(x, v, t) \log f(x, v, t) dx dv$$

is nonincreasing as a function of t for $t > 0$.

REFERENCE

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