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PERTURBATIONS OF THE UNILATERAL SHIFT

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Introduction. The study of the unilateral shift on a Hilbert space has been one of the most important subjects in operator theory, for it provides many useful examples and counterexamples to all parts of Hilbert space theory (see Halmos [**1**]). The purpose of this note is to announce our results concerning perturbations and similarity of the unilateral shift in a slightly general setting. To be precise, let S be an isometry on a separable Hilbert space H. We were able to show that $S+P$ is similar to S for a large class $\mathfrak{R}$ of S-admissible bounded linear operators P on H. To each $P\in\mathfrak{R}$, we constructed explicitly a nonsingular bounded linear operator W on H, such that $S+P=WSW^{-1}$. In particular, the unilateral shift S on l^2 of square-summable sequences, which sends $(x_0, x_1, x_2, \cdots)$ into $(0, x_0, x_1, x_2, \cdots)$ is similar to $S+\mu P$ for all infinite matrices $P=(p_{nm})$ with $\sum |p_{nm}| < \infty$ $(n, m=0, 1, 2, \cdots \infty)$ and all sufficiently small complex parameters μ. This result becomes interesting when it is compared with that of [**2**], where $P=(p_{nm})$ are required to be strictly lower-triangular and $p_{n+1,n}\neq -1$ for all n, in addition to the assumption that $\sum |p_{nm}| < \infty$.

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1. S-**admissible operators.** Throughout this note, let H, H' be separable Hilbert spaces and $Y=l^2(0, \infty; H')$. We denote by $\mathfrak{B}(H, H')$ the space of all bounded linear operators on H to H'. We write $\mathfrak{B}(H)$ for $\mathfrak{B}(H, H)$. The symbol $\langle\ ,\ \rangle$ stands for the inner products in H and H'.

DEFINITION 1.1. Let S be an isometry on H and $A\in\mathfrak{B}(H, H')$. The operator A is said to be S-smooth, if there exists a constant $M < \infty$ such that

$$(1)\qquad \sum_{n=0}^{\infty} \|AS^n u\|^2 \leqq M^2\|u\|^2 \quad\text{and}\quad \sum_{n=0}^{\infty} \|AS^{*n}u\|^2 \leqq M^2\|u\|^2,$$

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for all $u \in H$. In this case, we set $|A|_S$ to be the infimum of all constants M satisfying (1).

To each isometry S on H and operators $A, B \in \mathfrak{B}(H, H')$, we let

$$D(A, B, S) = \left\{ f \in Y : \sum_{n=1}^{\infty} \left\| \sum_{r=1}^{n} AS^{n-r}B^*f(r-1) \right\|^2 < \infty \right\},$$

and define

$$\begin{aligned} T(A, B, S)f(n) &= 0, \qquad && n = 0; \\ &= \sum_{r=1}^{n} AS^{n-r}B^*f(r-1), \qquad && n \geqq 1, \text{ for } f \in D(A, B, S). \end{aligned}$$

$T(A, B, S)$ is clearly a linear operator from $D(A, B, S)$ into Y. This operator $T(A, B, S)$ is a discrete version of the convolution operator C defined in [**3**].

DEFINITION 1.2. A class $\mathfrak{R}(S) \subset \mathfrak{B}(H)$ of operators is called a S-admissible class, if each $P \in \mathfrak{R}(S)$ can be written as the product $P = B^*A$ of two operators $A, B \in \mathfrak{B}(H, H')$, satisfying the following properties:

(i) both A and B are S-smooth,

(ii) the operator $T(A, B, S)$ is a bounded linear operator on Y into Y.

To each $P \in \mathfrak{R}(S)$, we set

$$[\![P]\!] = \max\left(\|T(A, B, S)\|,\ |A|_S \cdot |B|_S\right), \tag{2}$$

where $\|T(A, B, S)\|$ stands for its operator norm in $\mathfrak{B}(Y)$.

2. The main results.

THEOREM. *Let S be an isometry on H. Then for each $P = B^*A$ belongs to $\mathfrak{R}(S)$, and for any complex number μ with $|\mu| < 1/2[\![P]\!]$, the operator $S + \mu P$ is similar to S. The operator $W(\mu)$, given by the following formula,*

$$\langle W(\mu)u, v\rangle = \langle u, v\rangle + \mu \sum_{n=0}^{\infty} \langle AS^{*n+1}u, B(S + \mu P)^{*n}v\rangle, \qquad u, v \in H,$$

is a nonsingular bounded linear operator on H which implements the similarity, i.e. we have $(S+\mu P)W(\mu) = W(\mu)S$.

EXAMPLE. Let $H = l^2(0, \infty)$, S = the unilateral shift on H and $P = (p_{nm})$ be an infinite matrix with $|P| = \sum_{n,m=0}^{\infty} |p_{nm}| < \infty$. Let $H' = l^2((0, \infty) \times (0, \infty))$. Define A, B from H to H' as follows:

$$Ax_{nm} = |p_{nm}|^{1/2}x_m,$$

and

$$Bx_{nm} = \alpha_{nm}|p_{nm}|^{1/2}x_n, \quad \text{for all } x \in l^2(0, \infty),$$

where

$$\alpha_{nm} = p_{nm}/|p_{nm}| \qquad \text{for } p_{nm} \neq 0 \quad \text{and} \quad \alpha_{nm} = 0 \quad \text{when} \quad p_{nm} = 0.$$

It can be verified easily that A, $B \in \mathfrak{B}(H, H')$ with $B^*A = P$. We were able to show that A, B are S-smooth with $|A|_S$ and $|B|_S$ both bounded by $|P|^{1/2}$. And the operator $T(A, B, S)$ belongs to $\mathfrak{B}(Y)$ with its norm bounded by $|P|$. Consequently, every infinite matrix $P = (p_{nm})$ with $|P| = \sum_{n,m=0}^{\infty} |p_{nm}| < \infty$ is S-admissible with $[\![P]\!] \leq |P|$. We therefore have the following:

COROLLARY. *Let S be the unilateral shift on $l^2(0, \infty)$. If $P = (p_{nm})$ is an infinite matrix with $|P| = \sum_{n,m=0}^{\infty} |p_{nm}| < \infty$, then for all complex numbers μ with $|\mu| < 1/2|p|$, $S + \mu P$ and S are similar operators on $l^2(0, \infty)$.*

3. **Outline of proof.** To each S-admissible operator $P = B^*A$, we need to prove that there exists a constant $K < \infty$ such that

$$\sum_{n=0}^{\infty} \|B(S + \mu P)^{*n}u\|^2 \leq K\|u\|^2, \qquad \text{for all } u \in H. \tag{3}$$

This is the main and the most difficult part of the entire proof. There seems to be no easy way to get a reasonable estimate for $\|B(S+\mu P)^{*n}u\|$, and moreover S, P do not necessarily commute. To get around these difficulties, we first constructed a family $U(n, \mu) \in \mathfrak{B}(H)$ ($n = 0, 1, 2, \cdots$) having the property

$$\left(\sum_{n=0}^{\infty} \|BU(n, \mu)^*u\|^2\right)^{1/2} \leq |B|_S(1 - |\mu| [\![P]\!])^{-1}\|u\|.$$

Next we show that $(S+\mu P)U(n, \mu) = U(n+1, \mu)$ for all n. It follows that $U(n, \mu) = (S+\mu P)^n$, hence the inequality (3). Once the inequality (3) is obtained, we write $W(\mu) = 1 + \mu Q(\mu)$, with

$$\|Q(\mu)\| \leq |A|_S|B|_S(1 - |\mu| [\![P]\!])^{-1}.$$

Nonsingularity of $W(\mu)$ follows when we chose μ sufficiently small to make $\|\mu Q(\mu)\| < 1$. Finally $(S+\mu P)W(\mu) = W(\mu)S$ is established by a routine manipulation and observing that $S^*S = 1$.

REMARK. If S is unitary on H, then our Theorem holds for all parameters μ with $|\mu| < 1/\|P\|$ and we have, for $u, v \in H$,

$$\langle W(\mu)^{-1}u, v\rangle = \langle u, v\rangle + \mu \sum_{n=0}^{\infty} \langle AU(n, \mu)u, BS^{n+1}v\rangle.$$

Generalization to $l^p(0, \infty)$ for perturbations of the unilateral shift S, together with full details for the above results will appear elsewhere.

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