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GENERATION OF EQUICONTINUOUS SEMIGROUPS BY HERMITIAN AND SECTORIAL OPERATORS. I

BY ROBERT T. MOORE[1]

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1. **Introduction.** This announcement and its sequel [8] conclude a series, beginning with [6] and [7], in which classical Banach-algebraic techniques are adapted to treat problems in operator theory on a general locally convex space (lcs). These announcements provide a representative sampling of the results which will appear, with full proofs and examples, in [9].

The material discussed below is concerned with the more geometrical aspects of the generation and perturbation theory of continuous (and/or holomorphic) semigroups of operators on an lcs. It unifies and extends several earlier lines of development: the Hille-Yosida generation theory and Phillips perturbation theory for C_0 semigroups on B-spaces [1], the norm-estimate theory of holomorphic semigroups on B-spaces developed by Hille and Phillips [1], and the geometric theory of hermitian (selfadjoint) and sectorial generators of unitary groups and holomorphic semigroups respectively (due respectively to Stone [10] and Kato [2] for Hilbert spaces, and to Lumer-Phillips [3] and the author [5] for B-spaces). Applications are illustrated in §5 in the nonclassical setting of various differential operators and evolution equations on test function spaces.

The setting of the present announcement is that of a "calibrated" lcs $\mathfrak{X}$ with a "Lumer geometry" as described in [6] and [7]. By contrast, [8] is concerned with the more topological aspects of the theory, relating earlier work of Schwartz, Yosida, and Komatsu (see [11]) and the theory of "distribution semigroups" to the results discussed below. A primitive version of some of this material was sketched in [5], and references 6 and 7 cited there have been absorbed into the monograph [9].

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2. **Sectorial operators and holomorphic semigroups.** Here $\mathfrak{X}$ will denote a complex lcs, assumed throughout to be *complete* (some results are true in greater generality). A *calibration* Γ for $\mathfrak{X}$ is a collection of seminorms p which together induce the topology of $\mathfrak{X}$. For each angle $0 \leqq \Phi \leqq \pi/2$, we regard the symmetric closed sector

$$S_\Phi = \{z \in \mathbf{C}: |\arg z| \leqq \Phi\}$$

as an additive topological semigroup with the usual topology.

DEFINITION 1. (a) A family $\{T_z : z \in S_\Phi\} \subset \mathcal{L}(\mathfrak{X})$ of continuous linear operators is a *holomorphic semigroup of class* Φ or is in $H(\Phi)$, iff

(i) the map $z \to T_z$ is a semigroup homomorphism into the multiplicative semigroup of $\mathcal{L}(\mathfrak{X})$ that carries 0 into I and is *weak-operator continuous*, and

(ii) on the interior $\operatorname{int}(S_\Phi)$ the map is *weak-operator holomorphic.* (That is, for all $u \in \mathfrak{X}$ and u_* in the dual $\mathfrak{X}^*$, the map $z \to \langle T_z u, u_* \rangle \in \mathbf{C}$ is respectively continuous or holomorphic on the sets specified.)

(b) If for some calibration Γ, every T_z ($z \in S_\Phi$) is a *contraction* ($p(T_z u) \leqq p(u)$ for all $p \in \Gamma$), the semigroup is in $CH(\Phi, \Gamma)$.

REMARK. Essentially the same argument as that used in Chapter 7 of [5] shows that every $CH(\Phi, \Gamma)$ semigroup is "simply continuous" or "strongly continuous" ($z \to T_z u$ is continuous from S_Φ into $\mathfrak{X}$ for every $u \in \mathfrak{X}$). In the degenerate case $\Phi = 0$, a $CH(0, \Gamma)$ semigroup turns out to be exactly a C_0 equicontinuous semigroup in the sense of Yosida [11]. (One uses Theorem 4 of [6].)

DEFINITION 2. (a) Suppose that $\{T_z : z \in S_\Phi\}$ is in $H(\Phi)$. Then $u \in \mathfrak{X}$ is in the domain $D(A)$ of the *(infinitesimal) generator* A of the semigroup iff there exists a vector $Au \in \mathfrak{X}$ such that

$$\lim\{t^{-1}(T_t u - u) : t \to 0 \text{ in } [0, \infty)\} = Au;$$

then we say that A *generates* the semigroup.

(b) The semigroup is *smooth* iff $D(A) = \mathfrak{X}$.

If $\mathfrak{X}$ is barreled, then the generator A of a smooth semigroup is continuous. In the B-space setting, such "bounded" generators are very special, but smooth semigroups are typical of problems formulated on the proper locally convex space.

Generators of $CH(\Phi, \Gamma)$ semigroups can be characterized in terms of their *numerical ranges* as defined in [7]; we recall the main ideas here. If Γ calibrates $\mathfrak{X}$, then the $\mathfrak{X}$-part Λ_0 of a *Lumer geometry for* $(\mathfrak{X}, \Gamma)$ is defined by selecting a suitable *semi-inner product* $[\ ,\ ]_p$ for each $p \in \Gamma$ and setting $\Lambda_0 = \{[\ ,\ ]_p : p \in \Gamma\}$. Then the $\mathfrak{X}$*-part* of the *numerical range* of A is defined to be

$$W(A, \Lambda_0) = \{[Au, u]_p : (u, p) \in \mathfrak{X} \times \Gamma \quad \text{and} \quad p(u) = 1\}.$$

Using the natural dual calibration Γ_* for $\mathfrak{X}^*$, one similarly defines an $\mathfrak{X}^*$-*part* $\Lambda_1 = \{[\ ,\ ]_q : q \in \Gamma_*\}$ of a Lumer geometry and an $\mathfrak{X}^*$-*part* $W(A^\odot, \Lambda_1)$ of the numerical range, where $A^\odot$ denotes the *Phillips adjoint* of A. Then the *Lumer geometry* is $\Lambda_* = \Lambda_0 \cup \Lambda_1$, and Λ_*-*numerical range* of A is $W(A, \Lambda_*) = W(A, \Lambda_0) \cup W(A^\odot, \Lambda_1)$.

DEFINITION 3. If $0 \leqq \Phi \leqq \pi/2$, let

$$\Delta_\Phi = \{z \in C : \pi/2 + \Phi \leqq \arg z \leqq 3\pi/2 - \Phi\}$$

be the sector "orthogonal to" S_Φ (in the sense that their boundaries are perpendicular). Then A is Φ-*sectorial* with respect to a geometry Λ_* for $(\mathfrak{X}, \Gamma)$ iff $W(A, \Lambda_*) \subset \Delta_\Phi$.

THEOREM 1. *Let* $0 \leqq \Phi \leqq \pi/2$. *Then the following conditions on an operator* A *on a calibrated lcs* $(\mathfrak{X}, \Gamma)$ *are equivalent.*

(a) *The domain* $D(A)$ *is dense and* A *is* Φ-*sectorial with respect to some geometry* Λ_* *for* $(\mathfrak{X}, \Gamma)$.

(b) *The closure* $\overline{A}$ *of* A *is a densely defined* (*dd*) *operator which is* Φ-*sectorial with respect to every* Λ_* *for* $(\mathfrak{X}, \Gamma)$.

(c) *The closure* $\overline{A}$ *is a dd operator with* Γ-*spectrum* $\sigma_\Gamma(\overline{A}) \subset \Delta_\Phi$ *and if* $d_\lambda = \operatorname{dist}(\lambda, \Delta_\Phi) > 0$ *then* $d_\lambda \|(\lambda - \overline{A})^{-1}\|_\Gamma \leqq 1$.

(d) *The closure* $\overline{A}$ *generates a* $CH(\Phi, \Gamma)$ *semigroup* $\{T_z : z \in S_\Phi\}$.

(e) (*Equivalent only when* $\Phi > 0$.) *There exists a semigroup homomorphism* T *of* $\operatorname{int}(S_\Phi)$ *into the unit ball of the Banach algebra* $\mathfrak{F}_\Gamma(\mathfrak{X})$ *such that*

(i) $D(A) \supset \bigcup \{T_z \mathfrak{X} : z \in \operatorname{int}(S_\Phi)\}$ *and the latter is dense in* $\mathfrak{X}$, *and*

(ii) *the function* $z \to T_z$ *is complex-differentiable on* $\operatorname{int}(S_\Phi)$ *in the* $\|\cdot\|_\Gamma$ *sense, with* $(d/dz)T_z = AT_z$, *and admits a* $\|\cdot\|_\Gamma$-*convergent power series expansion about each* $z \in S_\Phi$.

CONVENTION. In view of the equivalence of (a) and (b) above, we will simply speak of A as "Φ-*sectorial with respect to* Γ," regarding the arbitrary choice of some geometry Λ_* as implicit.

The methods of proof are quite classical. For example, (a) and (b) are proved equivalent by means of Theorem 3 in [**7**] coupled with the method of Lemmas 3.3 and 3.4 in Lumer-Phillips [**3**]. Modulo [**7**], the remaining proofs can be built up from the B-space prototypes simply by using $\|\cdot\|_\Gamma$ for the "operator norm".

We recall that $\mathfrak{X}^\odot$ is the strong ($\beta(\mathfrak{X}^*, \mathfrak{X})$) closure of the domain $D(A^*)$ of the adjoint of A in $\mathfrak{X}^*$ [**7**]. Let $\Gamma_\odot$ be the calibration on $\mathfrak{X}^\odot$ obtained by restriction from Γ_*.

THEOREM 2. *Suppose* A *satisfies any one of the equivalent conditions in Theorem* 1. *Then the Phillips adjoint* $A^\odot$ *generates a* $CH(\Phi, \Gamma_\odot)$

semigroup on $\mathfrak{X}^{\odot}$ *whose values are just the restrictions of the adjoint semigroups* $\{T_z^*: z \in S_\Phi\}$ *to* $\mathfrak{X}^{\odot}$. *Consequently for any geometry* Λ_* *for* $(\mathfrak{X}^{\odot}, \Gamma_{\odot})$, $A^{\odot}$ *is* Φ*-sectorial.*

In some applications, especially those involving nonsmooth semigroups where $D(A^{\odot})$ is difficult to determine, direct calculation of the $\mathfrak{X}^*$-part $W(A^{\odot}, \Lambda_1)$ of the numerical range is highly impractical, and the following lemma is a more efficient test for Φ-sectoriality.

LEMMA 3. *A densely-defined operator* A *is* Φ*-sectorial with respect to* Γ *if it satisfies the two conditions*:

(a) *the* $\mathfrak{X}$*-part* $W(A, \Lambda_0) \subset \Delta_\Phi$ *for some Lumer geometry* Λ_* *for* $(\mathfrak{X}, \Gamma)$, *and*

(b) *the range* $(\lambda_0 - A)D(A)$ *is dense in* $\mathfrak{X}$ *for some* $\lambda_0 \notin \Delta_\Phi$.

3. Hermitian operators and Stone's theorem.

DEFINITION 4. A family $\{T_t: t \in \mathbf{R}\} \subset \mathcal{L}(\mathfrak{X})$ is a (weakly) C_0 *group* iff the families $\{T_t^+ = T_t: t \in [0, \infty)\}$ and $\{T_t^- = T_{-t}: t \in [0, \infty)\}$ are $H(0)$ semigroups such that $T_t^+ T_t^- = T_t T_{-t} = I$ for all $t \in [0, \infty)$. If for some calibration Γ for $\mathfrak{X}$ every T_t is a Γ-symmetry [6], the group is a *generalized unitary group* or is in $RC_0(\Gamma)$.

DEFINITION 5. (a) The infinitesimal generator A of a C_0 group is that of the positive semigroup $\{T_t^+ = T_t: t \in [0, \infty)\}$, and the group is smooth if $D(A) = \mathfrak{X}$.

(b) An operator H is hermitian with respect to a geometry Λ_* for $(\mathfrak{X}, \Gamma)$ iff $W(H, \Lambda_*) \subset \mathbf{R}$.

THEOREM 4. *Let* H *be an operator on a calibrated lcs* $(\mathfrak{X}, \Gamma)$. *Then the following are equivalent.*

(a) *The domain* $D(H)$ *is dense in* $\mathfrak{X}$ *and* H *is hermitian with respect to some geometry* Λ_* *for* $(\mathfrak{X}, \Gamma)$.

(b) *The closure* $\overline{H}$ *of* H *is a dd hermitian operator with respect to every* Λ_* *for* $(\mathfrak{X}, \Gamma)$.

(c) *The closure* $\overline{H}$ *is a dd operator with* $\sigma_\Gamma(H) \subset \mathbf{R}$, *and if* $\operatorname{Im}(\lambda) \neq 0$, $|\operatorname{Im}(\lambda)| \, \|(\lambda - H)^{-1}\|_\Gamma \leq 1$.

(d) *The closure* $A = (iH)^- = i\overline{H}$ *is the generator of a generalized unitary group.*

The theorem follows easily from Theorem 1. (Notice that H is hermitian iff $\pm iH$ are 0-sectorial.) In the same way, one obtains the obvious analog of Theorem 2 for dual groups. An analog of Lemma 3 is also available.

Generalized unitary groups sometimes arise as boundary values of $CH(\pi/2, \Gamma)$ semigroups. (If A is $\pi/2$-sectorial, $H = -iA$ is hermitian.)

4. Perturbation theorems. In the common setting of a reflexive calibrated lcs $(\mathfrak{X}, \Gamma)$, let us equip $\mathfrak{L}(\mathfrak{X})$ with the topology of simple convergence to form $\mathfrak{L}_s(\mathfrak{X})$, and pick out the topological subspace

$$\mathcal{O}_\Phi(\Gamma) = \{A \in \mathfrak{L}_s(\mathfrak{X}) \colon A \text{ is } \Phi\text{-sectorial with respect to } \Gamma\}$$

defined by $0 \leqq \Phi \leqq \pi/2$. By our remarks after Definition 1, the set $CH(\Phi, \Gamma)$ of holomorphic contraction semigroups can be viewed as a subset of $C(S_\Phi, \mathfrak{L}_s(\mathfrak{X}))$, the continuous functions from S_Φ to $\mathfrak{L}_s(\mathfrak{X})$, and can be endowed with the compact convergence topology.

THEOREM 5. (a) *The set* $\mathcal{O}_\Phi(\Gamma)$ *is a closed cone in* $\mathfrak{L}_s(\mathfrak{X})$.

(b) *The map* $A \to \{\text{“}\exp zA\text{''} \colon z \in S_\Phi\} = \{T_z \colon z \in S_\Phi\}$ *from generators to semigroups is continuous from* $\mathcal{O}_\Phi(\Gamma)$ *to* $CH(\Phi, \Gamma) \subset C(S_\Phi, \mathfrak{L}_s(\mathfrak{X}))$.

A comparable theorem is true for generalized unitary groups: the map from the real closed lcs of hermitians to groups-as-functions-on $\boldsymbol{R}$ is continuous.

5. **Examples.** Let $\mathfrak{X}$ be the space of C^∞ functions on R^n with period 1 along each coordinate axis (alias C^∞ functions on the n-torus). For each $1 \leqq r \leqq \infty$, we calibrate $\mathfrak{X}$ with a “topology of L^r-convergence of derivatives”, where $\|u\|_r = (\int_K |u|^r\,dx)^{1/r}$ for $1 \leqq r < \infty$ and $\|u\|_\infty = \sup\{|u(x)| \colon x \in K\}$ with K the unit cube. Letting $D_j = \partial/\partial x_j$, define

$$p_{r,m}(u) = \sum \left\{\left\|D_1^{\alpha(1)} \cdots D_n^{\alpha(n)} \cdots u\right\|_r^2 \colon \alpha(1) + \cdots + \alpha(n) \leqq m\right\}^{1/2}$$

and

$$\Gamma(r) = \{p_{r,m} \colon m = 0, 1 \cdots\}.$$

By a generalized Sobolev lemma, the $\Gamma(r)$ all calibrate the same nuclear Fréchet “test function” topology on $\mathfrak{X}$, with $r = 1$, 2 and ∞ of primary interest.

EXAMPLE 1. For every $1 \leqq r \leqq \infty$, the continuous operators $iD_j \in \mathfrak{L}(\mathfrak{X})$ can be shown by direct calculation to be $\Gamma(r)$-hermitian. As in Theorem 4 and the remark following Theorem 5, every $D = \sum \alpha_j D_j$ generates a smooth generalized unitary group for $\Gamma(r)$ which translates functions in the direction $(\alpha_i, \cdots, \alpha_n)$, and this group varies continuously with respect to the real parameters α_i.

EXAMPLE 2. Various methods (e.g. §6 of [8]) can be used to check that each D_j^2 is at least 0-sectorial (dissipative) for every $\Gamma(r)$, $1 \leqq r \leqq \infty$. For $r = 1$ and $r = \infty$, the numerical ranges can be shown to exhaust the entire left half-plane, while by contrast every D_j^2 is $\pi/2$-sectorial with respect to $\Gamma(2)$, exhibiting the extreme sensitivity of the numerical ranges to choice of calibration. Thus for $\Gamma(2)$ (which has a unique $\mathfrak{X}$-part of a Lumer geometry consisting of inner products) every nonnegative combination $L=\sum \alpha_j D_j^2$ generates a smooth holomorphic contraction semigroup in the right half-plane, with generalized unitary boundary group. The choice $\alpha_j=1$, $1 \leqq j \leqq n$, yields the solution to the heat equation on the n-torus, without recourse to ellipticity or "deficiency" arguments. Results for $r \neq 1$, 2, ∞ can be obtained by interpolation.

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University of Washington, Seattle, Washington 98105