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FIXED POINT SCHEMES

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Let S be a scheme and let G be a group scheme over S. If $\alpha: G\times X \to X$ is an action of G on X over S (cf. [4]), we say that (X, α)—or simply X—is a G-scheme over S. The 'fixed point functor' h_X^G of G in X is defined as follows. For each S-scheme Y, let Y_G denote the trivial G-scheme (Y, p_2). Then

$$h_X^G(Y) = (\text{set of } G\text{-linear } S\text{-morphisms } \varphi\colon Y_G \to X).$$

THEOREM 1. *If $\mathcal{C}$ is the category of locally noetherian S-schemes and quasicompact S-morphisms, X is a G-scheme in $\mathcal{C}$, and G is flat over S, then h_X^G is represented by a closed subscheme X^G of X.*

In this vast generality it is not to be expected that much detailed information about X^G can be obtained. Nevertheless, one does have the following 'rigidity' result when G is an abelian scheme over S (cf. [4]).

THEOREM 2. *Let G be an abelian scheme over S and let X be a connected locally noetherian G-scheme over S. Then either X^G is empty or $X^G=X$.*

It is conceivable that this property could be used as the starting point for the general theory of abelian schemes, e.g., commutativity and Chow's theorem (cf. [3]) are easy consequences of Theorem 2.

For a deeper study of fixed point schemes, we restrict ourselves to the category of algebraic schemes over a field k, acted upon by algebraic groups (i.e., smooth group schemes of finite type) over k. One result, which is related to a special case of a recent result of G. Horrocks [2], is

PROPOSITION 3. *Let G be a linear algebraic group over k. The largest k-closed normal subgroup H of G such that, for all proper connected G-schemes X over k, X^H is connected is the unipotent radical of G.*

For smooth schemes and 'very good groups' one has:

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PROPOSITION 4. *If a linearly reductive linear algebraic group G acts on a smooth algebraic scheme X over k, then X^G is smooth over k.*

It seems to be an open question whether X^G is smooth in the case of a semisimple group G acting on a smooth X over a field k of characteristic $p>0$. This is false for finite groups G such that p divides the order of G.

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