ON THE EQUATIONS $u_t+\nabla\cdot F(u)+0$ AND $u_t+\nabla\cdot F(u)=\nu\Delta u$[1]

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This paper presents several results on global solutions of the initial value problems for the first order nonlinear conservation law

$$u_t + \nabla\cdot F(u) = 0 \tag{1}$$

and the associated second order nonlinear parabolic equation

$$u_t + \nabla\cdot F(u) = \nu\Delta u, \qquad \nu > 0 \tag{2}$$

for an unknown scalar function $u=u(t, x)$ on the domain $D=\{(t, x)\in R^{d+1};\ t>0\}$. Here $F\in C^\infty(R^1, R^d)$. For both equations, the given initial data are

$$u(0, x) = u_0(x), \qquad x \in R^d. \tag{3}$$

We call these initial value problems IVP_1 and IVP_2 respectively. They are of interest as simplified prototypes of the initial value problems of gas dynamics (nonviscous and viscous respectively—cf. [2]).

We deal with weak solutions of IVP_1 and IVP_2. If $u\in L_1^{\mathrm{loc}}(D)$, we say that u is a *weak solution of IVP_1* if for each $\phi\in C^1(R^{d+1})$ of compact support

$$\iint_D [u\phi_t + F(u)\cdot\nabla\phi]dxdt + \int_{R^d} u_0(x)\phi(0, x)dx = 0. \tag{4}$$

We say that u is a *weak solution of IVP_2* if for each $\phi\in C^2(R^{d+1})$ of compact support

$$\iint_D [u\phi_t + \nu u\Delta\phi + F(u)\cdot\nabla\phi]dxdt + \int_{R^d} u_0(x)\phi(0, x)dx = 0. \tag{5}$$

It is well known [2] that weak solutions of IVP_1 are discontinuous and nonunique. For solutions of bounded variation locally in D, Vol'pert [3] has given a supplementary condition, called an *entropy condition*, on the discontinuities of a solution which singles out a unique solution in this class. We call this the *entropy solution*; it exists whenever u_0 is bounded and has bounded variation locally in R^d [3].

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We assume throughout that u_0 is integrable on R^d and has bounded variation there. It follows that $|u_0| \leqq M$ for some M.

If $u(t, x)$ is a weak solution of IVP_2 which is locally essentially bounded and has as distribution gradient $\nabla u(t, x)$ a bounded measure on compact sets of D, we call it a *regular solution* of IVP_2.

THEOREM 1. *If $u(t, x)$ is a regular solution of IVP_2, then $u\in C^\infty(D)$, equation* (2) *is satisfied in the classical sense in D, and the initial condition* (3) *is satisfied in the sense that, for each $\phi\in C^0(R^{d+1})$ of compact support,*

$$\int_{R^d} u(t, x)\phi(t, x)dx \to \int_{R^d} u_0(x)\phi(0, x)dx \quad as \quad t\to 0.$$

THEOREM 2. *IVP_2 has at most one regular solution.*

We approach the questions of existence and properties of solutions of IVP_1 and IVP_2 through a finite difference scheme used by Conway and Smoller [1] to solve IVP_1. Our methods are slightly stronger than those of [1], and permit simultaneous consideration of IVP_1 and IVP_2.

Let $h, q>0$ be mesh lengths. Let G be the d-dimensional lattice $G=\{x\in R^d;\ x=q\alpha \text{ for } \alpha\in Z^d\}$. We label points in G by their multi-indices α. Let $\delta(i)$ be the multi-index with 1 in the ith component and 0 in all others. If u is a function on G, u^α denotes its value at $\alpha\in G$. We consider maps $k\to u^\alpha(k)$ from the nonnegative integers to functions on G. Then our finite difference scheme may be written

$$\begin{aligned} h^{-1}\Big[u^\alpha(k+1) - (2d)^{-1}\sum_{i=1}^{d}(u^{\alpha+\delta(i)}(k) + u^{\alpha-\delta(i)}(k))\Big]& \\ + \sum_{i=1}^{d}(2q)^{-1}[F_i(u^{\alpha+\delta(i)}(k)) - F_i(u^{\alpha-\delta(i)}(k))] &= 0 \end{aligned} \tag{6}$$

with initial data

$$u^\alpha(0) = u_0^\alpha \tag{7}$$

where F_i is the ith component of F. It is clear that $u^\alpha(k)$ is uniquely determined through (6) by the initial data (7).

We may identify the $u^\alpha(k)$ (resp. u_0^α) with functions $U(t, x)$ (resp. $U_0(x)$) which are constant on "grid cells." It is with this identification in mind that we speak of convergence of solutions of (6) (resp. convergence of the initial data (7)) as $h, q\to 0$.

Let h_j, $q_j \to 0$ define a sequence of grids G_j. If $\|u_0^j\|_\infty \leqq M$, let $A = \max_i \sup_{|v| \leqq M} |F_i'(v)|$, and assume that the stability condition $Ad \leqq q_j/h_j$ holds for each G_j. It is possible to choose the initial data $U_0^j(x)$ so as to converge in $L_1(R^d)$ to u_0 with $\|U_0^j\|_\infty \leqq M$ and the total variation of ∇U_0^j bounded by that of ∇u_0. We assume below that such a choice has been made.

THEOREM 3. (i) *If $q_j^2/2dh_j \to \nu > 0$, then the finite difference solutions converge in $L_1^{\text{loc}}([0, T] \times R^d)$ for each fixed $T > 0$ to a regular solution of IVP_2.*

(ii) *If $q_j^2/2dh_j \to 0$, then there is a subsequence of G_j such that the finite difference solutions converge in $L_1^{\text{loc}}([0, T] \times R^d)$ for each fixed $T > 0$ to a weak solution of IVP_1.*

THEOREM 4. *Let $u_\nu(t, x)$ be regular solutions of IVP_2 (parameterized by ν), and let $u(t, x)$ be the entropy solution of IVP_1. Then $u_\nu(t, x) \to u(t, x)$ in $L_1^{\text{loc}}([0, T] \times R^d)$ as $\nu \to 0$ for each fixed $T > 0$.*

Some of these results were obtained in the author's doctoral dissertation at the Massachusetts Institute of Technology under the guidance of Professor James Glimm.

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