ON A CONJECTURE OF G. D. MOSTOW AND THE STRUCTURE OF SOLVMANIFOLDS

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Introduction. Let G be a connected solvable Lie group and let Γ be a closed subgroup of G. Then the quotient manifold G/Γ is called a solvmanifold. G. D. Mostow in a fundamental paper [6] proved

THEOREM 1. *Let G/C be a compact solvmanifold, let N be the nil-radical of G, and let Γ contain no nontrivial, connected subgroup normal in G. Then*

(a) *N contains the identity component of Γ,*

(b) *$N/N\cap\Gamma$ is compact,*

(c) *$N\Gamma$, the group generated by N and Γ in G, is closed, in G.*

Mostow has also conjectured the following:

MOSTOW CONJECTURE. A solvmanifold is a vector bundle over a compact solvmanifold.

In this paper we will announce results that yield a new proof of Theorem 1 and a proof of the Mostow Conjecture, as well as many of the known results on the structure of solvmanifolds as given in [1], [3] and [4] for instance. An outline of the proof of the Mostow Conjecture and the proof of Theorem 1 are given in §3.

1. Definitions and resume of known facts. Let N be a connected, simply connected nilpotent Lie group. A closed subgroup of N will be called a CN group. According to Malcev a CN group Δ can be characterized as a torsion free nilpotent group such that if Δ_0 is the identity component of Δ then Δ/Δ_0 is finitely generated. Further, if Δ is a CN group there exists a unique connected nilpotent Lie group Δ_R such that $\Delta_R\supset\Delta$ and Δ_R/Δ is compact. If Δ is a CN group with Δ_0 trivial we will call Δ an FN group.

In [3] and [6] it was shown that a group Γ is the fundamental group of a compact solvmanifold if and only if Γ satisfies an exact sequence

$$(1) \qquad 1\rightarrow\Delta\rightarrow\Gamma\rightarrow Z^s\rightarrow 1$$

where Δ is an FN group and Z^s denotes s copies of the integers. Fundamental groups of compact solvmanifolds will be called FS groups. If Δ in (1) is a CN group we will call Γ a CS group. If Γ is a CS group satisfying the exact sequence (1) there is a unique group Γ_R satisfying the exact diagram:

$$
\begin{array}{ccccccccc}
1 & \rightarrow & \Delta & \rightarrow & \Gamma & \rightarrow & Z^s & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \Delta_R & \rightarrow & \Gamma_R & \rightarrow & Z^s & \rightarrow & 1
\end{array} \tag{2}
$$

As it is convenient, we will often identify a connected simple connected nilpotent Lie group with its Lie algebra by the exponential mapping.

Now let G be a connected simply connected solvable Lie group. In [5] the semisimple splitting S of G was characterized as follows:

$$S = T\cdot M = T\cdot G$$

where M is the simply connected nil-radical of S and T is an abelian group of semisimple automorphisms of M, where S is generated by T and G.

In [7], Wang showed that if Δ is an FN group, then

$$1 \rightarrow \Delta_R \rightarrow \Gamma_R \rightarrow Z^s \rightarrow 1$$

has a semisimple splitting in the following sense: there exists an abelian group of automorphism of Γ_R such that:

(1) $T\cdot\Gamma_R = T\cdot\Delta^*$ where Δ^* is a CN group and $\Delta^*\supset\Delta_R$.

(2) T acts as a semisimple group of automorphisms of $\Delta_R{}^*$ such that $T(\Delta_R)=\Delta_R$ and T induces the trivial action on Δ_R^*/Δ_R.

2. Main theorems.

THEOREM 2. *Let Γ be a CS group satisfying the exact sequence* (1) *and let Γ_R satisfy diagram* (2). *There exists a closed subgroup Δ^* of Δ_R such that*

(a) *$\Delta^*\supset\Delta$ and Δ^*/Δ is finite,*

(b) *Δ^* is normalized by Γ.*

Further, if we let $\Gamma^*=\Gamma\Delta^*$, there exists a semisimple splitting $T\cdot\Gamma_R$ of Γ_R such that

(c) *Γ^* and Δ^* are invariant under T.*

DEFINITION. A group $T\cdot\Gamma^*$ satisfying the conclusion of Theorem 2 is called a semisimple splitting of Γ.

THEOREM 3. *Let Γ be a $C\cdot S$ group and let Γ_1^* and Γ_2^* be two semisimple splittings of Γ. Then Γ_1^* and Γ_2^* are commensurable. Further, any automorphism of Γ has a unique extension to any semisimple splitting of Γ.*

LEMMA 4. *Let G be a connected, simple connected solvable Lie group and let Γ be a closed subgroup of G. Then Γ is a $C\cdot S$ group.*

THEOREM 5 (NIL-SHADOW). *Let G be a connected, simple connected solvable Lie group and let Γ be a closed subgroup of G. Let S be the semisimple splitting of S and let $\Gamma_S = T_\Gamma \cdot \Gamma^*$ be a semisimple splitting of Γ. There exists an isomorphism η: $\Gamma_S \to S$ such that $\eta(\Gamma_S)$ is a closed subgroup of S with $\eta(\Gamma) = \Gamma$. Further, there exists a semidirect product presentation of $S = T \cdot M$ such that $\eta(T_\Gamma) \subset T$ and if $\Gamma_S = T_\Gamma \cdot \Delta^*$ then $\Delta^* \subset M$ as a closed subgroup, where Δ^* is a CN group.*

3. **Applications.** In order to show the power of the nil-shadow theorem, we will assume it and prove Theorem 1 and outline a proof of the Mostow Conjecture.

DEFINITION. Let N be a connected, simply connected nilpotent Lie group and let B be a group of automorphisms of N. We define an action of $B \cdot N$ on N, called the affine action, as follows:

Let $(b, n) \in B \cdot N$ and let $m \in N$. We define

$$(b, n)(m) = b^{-1}(m)n$$

and if $\xi \in B \cdot N$ we denote $a(\xi)$ as the affine action of ξ on N.

We begin by stating, without proof, the following lemma.

LEMMA 6. *Let G be a connected, simply connected solvable Lie group and let Γ be a closed subgroup of G. Let $S = T \cdot M$ be the semisimple splitting of G. Then if we consider $\Gamma \subset S$ acting on M by the affine action then $M/a(\Gamma)$ is diffeomorphic to S/Γ.*

OUTLINE OF MOSTOW'S CONJECTURE. Let $\Gamma_S = T_\Gamma \cdot \Delta^*$ be a semisimple splitting of Γ with Δ^* in M and $T_\Gamma \subset T$. Then Δ_R^* is a subgroup of M which is invariant under Γ and $\Delta_R^*/a(\Gamma)$ is a compact solvmanifold. Let V be a vector space such that $M = V \oplus \Delta_R^*$ and such that $T_\Gamma(V) = V$. The existence of V follows from the fact that T_Γ leaves Δ_R^* invariant and that T_Γ acts semisimply on M. Hence every element of M may be written uniquely as $v \cdot \delta$, where $\delta \in \Delta_R^*$, $v \in V$. Further if $\gamma = (t, n)$, $n \in \Delta_R^*$, $t \in T_\Gamma$ we have

$$(t, n)v \cdot \delta = t(v) \cdot t(\delta)n = v'\delta'$$

where $v' \in V$ and $\delta' \in \Delta_R^*$. Thus we see that the images of the sets $V \cdot \delta$, $\delta \in \Delta_R^*$ gives a fiber bundle structure to $M/a(\Gamma)$ with $\Delta_R^*/a(\Gamma)$ as compact base space. It is easy to see that the action is linear and so that this is a vector bundle over the compact solvmanifold $\Delta_R^*/a(\Gamma)$.

Let us now prove Theorem 1. Assume now that all notation is as above and that in addition to this, G/Γ is compact. Then by the proof of the Mostow Conjecture we have that $\Delta_R^* = M$ or Δ^* is a closed cocompact subgroup of M. Thus, if Δ_0^* denotes the identity component of Δ^*, Δ_0^* is normal in M and invariant under T_Γ.

The following lemma is straightforward and its proof will be omitted.

LEMMA 7. *Let all notation be as above.* $\Gamma N/N$ *is a discrete subgroup of* G/N *if and only if* T_Γ *is a discrete subgroup of* T.

LEMMA 8. *Let all notation be as above. If* $a(M/\Delta_0^\#)$ *denotes the automorphism group of* $M/\Delta_0^\#$ *then the natural homomorphism* δ: $T_\Gamma \to a(M/\Delta_0^\#)$ *has trivial kernel.*

PROOF. Let $t \in T_\Gamma$ be in the kernel of δ. Then the range X of $(t-I)$, where I is the identity transformation, is in $\Delta_0^\#$. Since T is abelian, X is an invariant subspace of M under T. Hence the ideal, $\mathfrak{I}(X)$, generated by X in M is in $\Delta_0^\#$ and invariant under T. Thus $\mathfrak{I}(X)$ is an ideal in δ and so in G. This contradicts our hypothesis unless $X=0$ and the kernel of δ is trivial.

PROOF OF THEOREM 1. Since $\delta(T_\Gamma)$ preserves a discrete cocompact subgroup of $M/\Delta_0^\#$, $\delta(T_\Gamma)$ is a discrete subgroup of $a(M/\Delta_0^\#)$. Hence T_Γ is a discrete subset of $a(M)$. Lemma 8 now applies to complete the proof of Theorem 1.

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