# MAXIMAL RATES OF DECAY OF SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

BY WALTER LITTMAN

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It has been proved by C. Morawetz [2] that if $u(x, t)$ is a solution of the relativistic wave equation

$$u_{tt} - \Delta u + u = 0$$

for all $x=(x_1, x_2, \cdots, x_n)$ and $t$, having finite energy at $t=0$, and vanishing in the forward light cone $|x| < t$, $t > 0$, then it must vanish identically. On the other hand the author [1] has obtained a generalization of Rellich's Theorem (concerning decay of solutions of the reduced wave equation $\Delta u + u = 0$) to a class of (not necessarily elliptic) equations with constant coefficients of arbitrary order. The present note is intended to announce a number of results which are natural generalizations of and improvements of both aforementioned results. Detailed proofs will appear elsewhere.

Let $P(\xi) = P(\xi_1, \xi_2, \cdots, \xi_N)$ be a polynomial with real coefficients. Throughout, we make the following assumptions:

1. The real solution set $S$ of $P(\xi)=0$ is nonempty.
2. Grad $P(\xi) \neq 0$ in $S$, and hence $S$ is a smooth $N-1$ dimensional manifold.
3. The Gaussian curvature of $S$ never vanishes.

Assign a unit normal $\boldsymbol{n}$ to each point of $S$, varying continuously. The totality of all $\boldsymbol{n}$ fill an open set $\mathfrak{N}$ on the unit sphere, giving rise to an open cone $\mathcal{K}$ in $R^N$ in the sense that $\mathcal{K}$ consists of all $r\boldsymbol{n}$, $\boldsymbol{n} \in \mathfrak{N}$, $r \geqq 0$.

Define $\mathfrak{N}_\epsilon$ as that subset of $\mathfrak{N}$ consisting of points whose (spherical) distance to the boundary of $\mathfrak{N}$ exceeds $\epsilon$. $\mathcal{K}_\epsilon$ will denote the cone generated by $\mathfrak{N}_\epsilon$. $-\mathfrak{N}$ will denote the set of vectors $-\boldsymbol{n}$, with $\boldsymbol{n} \in \mathfrak{N}$, and similarly for $-\mathcal{K}$. $\mathfrak{N}'$ denotes the complement of $\mathfrak{N}$ on the unit sphere, and $\mathcal{K}'$ the corresponding cone. $\overline{\mathcal{K}}$ denotes the closure of $K$.

We will write

$$Lu \equiv P\left(\frac{1}{i}\frac{\partial}{\partial x_1}, \cdots, \frac{1}{i}\frac{\partial}{\partial x_N}\right)u \equiv P\left(\frac{1}{i}\frac{\partial}{\partial x}\right)u.$$

THEOREM I. *Suppose, under the foregoing Assumptions* 1–3, $u(x)$ *is a function satisfying*

$$P\left(\frac{1}{i}\frac{\partial}{\partial x}\right)u = 0 \qquad in\ R^N,$$

$$u = O(1 + |x|^\alpha) \qquad for\ some\ \alpha,$$

$$u = o(|x|^{(1-N)/2}) \qquad uniformly\ in\ every\ \mathcal{K}_\epsilon,$$

*then* $u\equiv 0$ *in* $R^N$.

Assumption 4. Each (complex) irreducible factor $P_j$ has an $N-1$ dimensional real solution set $S_j$.

THEOREM II. *Under Assumptions* 1–4, *if*

$$Lu = f\ has\ compact\ support,$$

$$u = O(1 + |x|^\alpha) \qquad for\ some\ \alpha,\ uniformly\ in\ R^N,$$

$$u = o(|x|^{(1-N)/2}) \qquad uniformly\ in\ every\ \mathcal{K}_\epsilon,\ -\mathcal{K}_\epsilon,$$

*then u has compact support.*

*Note.* For the equation $\Delta u+u=0$ the polynomial $P(\xi)=1-|\xi|^2$, and $S$ is the sphere $\xi_1^2+\xi_2^2+\cdots\xi_N^2=1$. $\mathcal{K}$, $\mathcal{K}_\epsilon$ are simply $R^N$, and we obtain Rellich's Theorem from Theorem II. On the other hand, for the equation $u_{tt}-\Delta u+u=0$, setting

$$t = \xi_N,\ P(\xi) = \xi_1^2 + \xi_2^2 + \cdots + \xi_{N-1}^2 - \xi_N^2 + 1,$$

and $S$ is the two sheeted hyperboloid

$$\xi_1^2 + \xi_2^2 + \xi_{N-1}^2 - \xi_N^2 + 1 = 0.$$

If we assign $\boldsymbol{n}$ so that it always points "up", i.e., its $N$th component is always positive, then $\mathcal{K}$ is the forward light cone, and applying Theorem I we obtain C. Morawetz's result, even a slight improvement. However, if we choose $\boldsymbol{n}$ to point "up" in the "upper" sheet and "down" in the "lower" sheet, $\mathcal{K}$ will be the union of the forward and backward light cones, and we get a worse result. Thus $\boldsymbol{n}$ has to be chosen with care.

ASSUMPTION 5. (a) $P((1/i)(\partial/\partial x))$ is hyperbolic with respect to $x_N$.
(b) $\overline{\mathcal{K}}$ does not intersect the set $x_N\leqq 0$ except at the origin.

THEOREM III. *Under Assumptions* 1–5, *suppose*
$Lu=0$ *for* $x_N>0$,
*the Cauchy data of u has compact support on* $x_N=0$,
$u(x_1, x_2\cdots x_{N-1}, 0)$ *has compact support in* $R^{N-1}$,
$u=O(1+|x|^\alpha)$ *uniformly for* $x_N>0$ *for some* $\alpha$,

*$u=o(|x|^{(1-N)/2})$ uniformly in some open cone $\mathcal{K}^*$ such that $\mathcal{K}^*\cap\mathcal{K}_j \neq 0$ $j=1, \cdots, r$ (where each $\mathcal{K}_j$ corresponds to the surface $S_j$ and $r$ is the number of these surfaces).*

*Then $u=0$ for $x_N>0$.*

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## BIBLIOGRAPHY

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UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455