RELATIVE HAUPTVERMUTUNG FOR NEIGHBORHOODS OF 1-FLAT SUBMANIFOLDS WITH CODIMENSION TWO

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1. Recently Kirby and Siebenmann have given general solutions of Hauptvermutung [5] and relative Hauptvermutung for neighborhoods of locally flat submanifolds [6]. In this note we announce some results about relative Hauptvermutung for neighborhoods of 1-flat submanifolds with codimension two (compare [11] and [3]).

We shall say that manifold pairs (Q, M) and (Q', M') are *topologically micro-equivalent*, if there are open neighborhoods U, U' of M, M' in Q, Q' and a homeomorphism $h: (U, M)\to(U', M')$, called a *topological micro-equivalence* between (Q, M) and (Q', M'). We shall say that PL manifold pairs (Q, M) and (Q', M') are PL *micro-equivalent*, if there are open neighborhoods V, V' of M, M' in Q, Q' and a PL homeomorphism $g: (V, M)\to(V', M')$, called a PL *micro-equivalence* between (Q, M) and (Q', M').

We shall prove the following

THEOREM A. *Let (Q, M) and (Q', M') be proper PL orientable* (4, 2)-*manifold pairs. Suppose that M is compact and that there is a topological micro-equivalence $h: (U, M)\to(U', M')$. Then there is a PL micro-equivalence $g: (V, M)\to(V', M')$. Further, if $h|M$ is already PL, then we can take g so that $g|M=h|M$.*

In order to extend this result to the higher dimensional case, we need some niceness condition for singularities.

Let (Q, M) be a proper PL $(m+2, m)$-manifold pair. We shall say that M is *locally flat at a point* x of M, if the link pair of x in (Q, M) is PL equivalent to the standard sphere or ball pair. If M is not locally flat at x, then x is called a *singular point* of M, and the link pair is called the *singularity*. We shall say that M is 1-*flat* in Q, if the set of singular points consists of isolated points. Note that if M is 1-flat in Q, the link pair of any point of M in (Q, M) is a locally flat PL $(m+1, m-1)$-sphere or ball pair; i.e. PL$(m-1)$-knot or disk knot, and that if M is of dimension two, then M is always 1-flat in Q. We

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shall say that a PL $(m-1)$-knot or disk knot is *nice*, if the complement of an open regular neighborhood of the embedded sphere or ball has a fundamental group isomorphic with that of the boundary.

THEOREM B. *Let* (Q, M) *and* (Q', M') *be proper orientable PL* $(m+2, m)$*-manifold pairs. Suppose that* M, M' *are compact,* 1*-flat in* Q, Q' *with nice singularities, and* $m \geqq 5$. *If there is a topological micro-equivalence* $h: (U, M) \to (U', M')$ *such that* $h|M$ *is PL, then there is a PL micro-equivalence* $g: (V, M) \to (V', M')$ *such that* $g|M = h|M$.

On the contrary we have the following remarkable counterexample.

THEOREM C. *For each even integer* $m \geqq 4$, *there are abstract regular neighborhoods* (N, K), (N', K') *of m-spheres* K, K' *with codimension two which are locally flat except for single points, resp., and an embedding* $h: (N, K) \to (N', K')$ *such that* $h|K$ *is PL, but there is no PL micro-equivalence between* (N, K) *and* (N', K').

This example ensures that even if the local flatness of a submanifold breaks at only one point, the relative Hauptvermutung for neighborhoods is false, which should be compared with the affirmative answer for the locally flat case by Kirby and Siebenmann [6].

2. Topological invariance problem of singularities. Let (Q, M) be a proper PL $(m+2, m)$-manifold pair. For a point x of M, by a *link pair* of x in (Q, M) we shall mean a pair $(lk(x, K), lk(x, L))$, written $lk(x; K, L)$, of links of x in K and L for some division (K, L) (or partition) of (Q, M). Then the PL homeomorphism class of the link pair $lk(x; K, L)$ does not depend on the choice of the division (K, L) of (Q, M) and will be denoted by $lk(x; Q, M)$. We examine the topological invariance of $lk(x; Q, M)$.

LEMMA 1. *Let* (Q, M) *and* (Q', M') *be proper PL* $(m+2, m)$*-manifold pairs. Suppose that there is a topological micro-equivalence* $h: (U, M) \to (U', M')$. *Then* $lk(x; Q, M)$ *and* $lk(x; Q', M')$ *are topologically invertible cobordant.*

PROOF. One can take divisions (K, L), (K_1, L_1) of (Q, M) and (K', L'), (K_1', L_1') of (Q', M') so that

$$c_1 = \mathrm{cl}(\mathrm{st}(hx; K', L') - h(\mathrm{st}(x; K, L))),$$

$$c = \mathrm{cl}(h(\mathrm{st}(x; K, L)) - \mathrm{st}(hx; K_1', L_1'))$$

and

$$c_2 = \mathrm{cl}(\mathrm{st}(hx; K_1', L_1') - h(\mathrm{st}(x; K_1, L_1))$$

are topological cobordisms of manifold pairs, where, for example.

st(x; K, L) denotes the star pair (st(x, K), st(x, L)). Since the composition $c_1 \circ c = \mathrm{cl}(\mathrm{st}(hx; K', L') - \mathrm{st}(hx; K_1', L_1'))$ *is* PL *equivalent to* $lk(hx; K', L') \times [0, 1]$ and $c \circ c_2 = \mathrm{cl}(h(\mathrm{st}(x; K, L) - \mathrm{st}(x; K_1, L_1)))$ is topologically equivalent to $lk(x; K, L) \times [0, 1]$, it follows that c is the required invertible cobordism between $lk(x; Q, M)$ and $lk(hx; Q', M')$, completing the proof.

Let (A, B) and (A', B') be proper PL manifold pairs. Let N and N' be derived neighborhoods of B and B' in A and A', resp. Putting $E = \mathrm{cl}(A - N)$ and $E' = \mathrm{cl}(A' - N')$, we shall call them *exteriors* of B and B' in A and A', resp.

LEMMA 2. *Suppose that* (A, B) *and* (A', B') *are topologically invertible cobordant. Then there is a map* $f: (A, B) \to (A', B')$ *such that* $f(E) = E'$, $f(\partial E) = \partial E'$, $f(N) = N'$ *and* $f| (E, \partial E): (E, \partial E) \to (E', \partial E')$ *is a homotopy equivalence.*

In fact, by the argument of Stallings [12, Theorem 2], $(A, B) \times R$ and $(A', B') \times R$ are homeomorphic. The composition $(A, B) \xrightarrow{\times 0} (A, B) \times R \to (A', B') \times R \to (A', B')$ can be deformed to the desired map, (refer to [2, proof of Lemma 5.1]), where the third map is the projection onto the first factor.

Here are examples of PL knots or disk knots for which topological invertible cobordism implies PL equivalence.

EXAMPLE 1. *Let* (A, B) *be a PL* m*-knot* (*or disk knot*) *topologically invertible cobordant to the standard one. Suppose that* $m \neq 2$ (*or* $m \neq 2, 3$). *Then* (A, B) *is PL equivalent to the standard one.*

This is an immediate consequence of the unknotting theorem.

EXAMPLE 2. *Let* (A, B) *and* (A', B') *be PL* 1*-knots* (*classical knots*) (*or disk knots*). *If* (A, B) *and* (A', B') *are topologically invertible cobordant, then they are PL equivalent.*

This follows from Waldhausen's Theorem [15]. In fact, Schoenflies Theorem and the existence of the minimal surface for a classical knot guarantees that an exterior of a classical knot is an irreducible and sufficiently large 3-manifold. By making use of Wall's surgery obstruction theory [14] (also see [9]) and the pseudo-isotopy classification theorem of PL homeomorphisms of $S^{m-1} \times S^1$ ([1] and [2]) we have a partial answer to the higher dimensional case:

EXAMPLE 3. *Let* (A, B) *and* (A', B') *be nice PL* $(m-1)$*-knots or disk knots. Suppose that* $m \geqq 5$. *If* (A, B) *and* (A', B') *are topologically invertible cobordant, then they are PL equivalent.*

The following counterexample is due to Siebenmann [10, p. 741].

COUNTEREXAMPLE (SIEBENMANN). *For each even integer* $m \geqq 4$, *there are topologically distinct PL $(m-1)$-knots (S, Σ) and (S', Σ') that are PL invertible cobordant.*

3. **Outline of the proof of theorems.** In [7] Noguchi defined the Euler class $\chi(Q, M)\in H^2(M, Z)$ for a second derived neighborhood of a compact orientable proper PL m-submanifold M in an orientable PL $(m+2)$-manifold Q. It is to be noted that if $m=2$, then by Takase [13] the Euler number $\langle\chi(Q, M), [M]\rangle$ is just the self-intersection number of the fundamental homology class (M) of M in Q; i.e. $\langle D(M), (M)\rangle$, where $[M]$ is the fundamental homology class of M and $D: H_2(Q)\to H^2(Q, \partial Q)$ is the Poincaré duality. If $m\geqq 3$ and if M is 1-flat with singular points $x_1, \cdots, x_n$, then $i^*\chi(Q, M)=\chi(Q_0, M_0)$, where $Q_0=Q-\bigcup_{i=1}^n \mathrm{st}(x_i, K)$, $M_0=M-\bigcup_{i=1}^n \mathrm{st}(x_i, L)$ for the first barycentric subdivision (K, L) of a division of (Q, M), $\chi(Q_0, M_0)$ is the Euler class of the normal bundle of M_0 in Q_0 and $i^*: H^2(M, Z)\to H^2(M_0, Z)$ is the monomorphism induced by the inclusion map $i: M_0\to M$.

Now we are ready to give an outline of the proof of theorems. Suppose that there is a topological micro-equivalence $h: (U, M)\to(U', M')$ and that $m\neq 3, 4$, and $h|M$ is already PL, if $m\geqq 5$. In case $m=2$, by Hauptvermutung for surfaces we may assume that $h|M$ is already PL. We will show that the PL homeomorphism $h|M$ can be extended to a PL micro-equivalence $g: (V, M)\to(V', M')$. For this in view of Noguchi's Theorem [8] we have only to show that $(h|M)^*\chi(Q', M')=\chi(Q, M)$ and $h|M$ preserves singular points so that $lk(hx_i; Q', M')=lk(x_i; Q, M)$ for all singular points x_i. First, by Example 1, $h|M$ preserves singular points; i.e. hx is a singular point of M' in Q' if and only if x is a singular point of M in Q and secondly, by Examples 2 and 3 $lk(hx_i; Q', M')=lk(x_i; Q, M)$ for all singular points x_i of M in Q. Thirdly notice that h can be deformed to a map whose restriction to the locally flat part is a fiber homotopy equivalence between normal bundles for the locally flat parts (Q_0, M_0) and (Q_0', M_0'). This together with the interpretation of $\chi(Q, M)$ given in the above shows that $(h|M)^*\chi(Q', M')=\chi(Q, M)$. Thus Noguchi's Theorem [8] completes the proof of Theorems A and B. In order to prove Theorem C, for each even integer $m\geqq 4$ we take PL $(m-1)$-knots (S, Σ) and (S', Σ') given in the counterexample above. Form PL $(m+2)$-manifolds $N=0*S\cup_{(\Sigma\times D^2)} D^m\times D^2$ and $N'=0*S'\cup_{(\Sigma'\times D^2)} D^m\times D^2$ from cones $0*S$ and $0'*S'$ by attaching handles of index m along collar neighborhoods $(\Sigma\times D^2)$ and $(\Sigma'\times D^2)$ of Σ and Σ' in S and S', resp. Then N and N' are abstract regular neighborhoods of $K=0*\Sigma\cup_\Sigma (D^m\times 0)$ and $K'=0'*\Sigma'\cup_{\Sigma'}(D^m\times 0)$

which have only single singularities (S, Σ) and (S', Σ'), resp. By the invertibility of the PL cobordism between (S, Σ) and (S', Σ'), we have a topological embedding $h: (N, K) \to (N', K')$ such that $h|K$ is a PL homeomorphism, (refer to [4, §5] and [11]). Hence (N, K) and (N', K') are topologically micro-equivalent. On the other hand, singularities (S, Σ) and (S', Σ') are distinct. Therefore, (N, K) and (N', K') are never PL micro-equivalent, completing the outline of the proof of Theorem C.

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