CROSS SECTIONALLY SIMPLE SPHERES

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Communicated by R. H. Bing, April 16, 1968

J. W. Alexander [1] suggested that a 2-sphere S in E^3 is tame if each horizontal cross section is either a point or a simple closed curve. It is not clear whether he presumed that his proof was valid for nonpolyhedral spheres, but his proof implies that there is a homeomorphism h of E^3 onto itself which is invariant on horizontal planes and which takes S onto a round 2-sphere. Bing [6] has described a nonpolyhedral 2-sphere S for which there is no such homeomorphism h.

In this paper we give a proof of Alexander's conjecture. The proof, however, is not elementary as it depends indirectly on Dehn's Lemma [8], Bing's Side Approximation Theorem [2], and Bing's Characterization of tame spheres with homeomorphic approximations in their complementary domains [4].

We assume that S lies exactly between the planes $z=1$ and $z=-1$ and we let $J_t=S\cap\{(x, y, z)\,|\,z=t\}$ be the horizontal cross section of S at the $z=t$ plane. Note that J_t is a simple closed curve for $-1<t<1$ and J_{-1}, J_1 are points. We let D_t be the disk J_t bounds in the $z=t$ plane. The ϵ-neighborhood of a set X is denoted by $N(X, \epsilon)$, Diam A is the diameter of A, and S^1 stands for the standard 1-sphere. If $-1<\alpha<\beta<1$ and h is a homeomorphism of $S^1\times[\alpha, \beta]$ into Int S such that

(1) $h(y\times[\alpha, \beta])$ is a vertical line segment for $y\in S^1$, and

(2) $h(S^1\times t)$ lies in the plane $z=t$ for $t\in[\alpha, \beta]$,

then $A(h, t)$ denotes the annulus in the $z=t$ plane bounded by $h(S^1\times t)$ and J_t, $S(\alpha, \beta)$ denotes the annulus $S\cap\{(x, y, z)\,|\,\alpha\leqq z\leqq\beta\}$ and $T(h)$ denotes the torus $h(S^1\times[\alpha, \beta])\cup A(h, \alpha)\cup A(h, \beta)\cup S(\alpha, \beta)$.

LEMMA 1. *If $t\in(-1, 1)$ and $\epsilon>0$ then there are rational numbers α and β and a homeomorphism h: $S^1\times[\alpha, \beta]\to$Int S such that*

(1) $-1<\alpha<t<\beta<1$,

(2) *$h(y\times[\alpha, \beta])$ is a vertical line segment for each $y\in S^1$,*

(3) *$h(S^1\times r)$ lies in the horizontal plane $z=r$ for $r\in[\alpha, \beta]$,*

(4) *$T(h)$ lies in an ϵ-neighborhood of J_t, and*

(5) *$h(S^1\times t)$ is homeomorphically within ϵ of J_t.*

PROOF. There is a simple closed curve J in the $z=t$ plane such that $J\subset$Int S, J is homeomorphically within ϵ of J_t, and the annulus A bounded by J and J_t in the $z=t$ plane lies in $N(J_t, \epsilon)$. J may be moved slightly in the vertical direction so there is a $\delta>0$ and a homeomorphism $g: S^1\times[t-\delta, t+\delta]\to\text{Int } S$ such that $g(y\times[t-\delta, t+\delta])$ is a vertical line segment for $y\in S^1$, $g(S^1\times r)$ lies in the $z=r$ plane for $r\in[t-\delta, t+\delta]$ and $g(S^1\times t)=J$. Since the annulus $A(g, t)$ lies in $N(J_t, \epsilon)$ and $\lim_{r\to t} A(g, r)=A(g, t)$, there are rational numbers α and β such that $t-\delta<\alpha<t<\beta<t+\delta$ and $A(g, r)\subset N(J_t, \epsilon)$ if $r\in[\alpha, \beta]$. Take $h=g|S^1\times[\alpha, \beta]$.

The following lemma is an easy consequence of the Tietze Extension Theorem and the fact that small subsets of S lie in small subdisks on S.

LEMMA 2. *If D is a 2-cell and $\epsilon>0$ then there is a $\delta>0$ such that if f is a map of* Bd D *into a δ-subset of* Int S *then f may be extended to D so that $f(D)$ lies in an ϵ-subset of* Cl(Int S).

To establish that S is tame from Int S we show that Int S is 1-ULC and use Bing's characterization of tame 2-spheres in E^3 [3]. That Int S is 1-ULC is an easy consequence of Lemmas 2 and 3.

LEMMA 3. *If f is a map of 2-cell D into* Cl(Int S) *such that* $f(\text{Bd } D)\subset\text{Int } S$ *and $\epsilon>0$ then there is a map $g: D\to$* Int S *such that* $f|\text{Bd } D=g|\text{Bd } D$ *and* $g(D)\subset N(f(D), \epsilon)$.

PROOF. The map f is adjusted in three steps to obtain g. In Step I, f is adjusted so that $f(D)$ misses the points J_1 and J_{-1}. In Step II, f is further adjusted so that $f(D)\cap S$ is 0-dimensional and $f(D)\cap J_r=\emptyset$ if r is a rational number. The map f is altered in Step III so that $f(D)\subset\text{Int } S$.

Step I. There is a $\delta>0$ such that Diam $D_{1-\delta}<\epsilon/3$, Diam $D_{-1+\delta}<\epsilon/3$, $D_{1-\delta}$ separates $f(\text{Bd } D)$ from J_1 on $f(D)$, and $D_{-1+\delta}$ separates $f(\text{Bd } D)$ from J_{-1} on $f(D)$. In two applications of the Tietze Extension Theorem, as indicated in [7, Lemma 1], we may adjust f to obtain a map $f_1: D\to\text{Cl(Int } S)$ such that $f|\text{Bd } D=f_1|\text{Bd } D$, $f_1(D)\subset N(f(D), \epsilon/3)$, $J_1\notin f_1(D)$, and $J_{-1}\notin f_1(D)$.

Step II. Since for $-1<r<1$, J_r is a tame simple closed curve, it follows from the techniques of [5] that there is a map $f_2: D\to\text{Cl(Int } S)$ such that $f_2|\text{BD } D=f_1|\text{Bd this } D$, $f_2(D)\subset N(f_1(D), \epsilon/3)$, $f_2(D)\cap S$ is 0-dimensional, and $J_r\cap f_2(D)=\emptyset$ if r is rational.

Step III. For $-1<t<1$ there is a $\delta_t>0$ such that if J is homeomorphically within δ_t of J_t then each δ_t subset of J lies in an $\epsilon/18$ arc in J. Since $f_2(D)\cap S$ is compact and 0-dimensional, for each $-1<t<1$ there is a collection $\mathcal{C}_t$ of disjoint open sets in E^3 covering $f_2(D)\cap S$ such that if $A\in\mathcal{C}_t$ then Diam $A<\delta_t$ and $A\cap f(\text{Bd } D)=\emptyset$. Let λ_t be a positive number which is less than the distance between $f_2(D)-\bigcup\mathcal{C}_t$

and J_t and which is less than $\epsilon/36$. By Lemma 1 there are rational numbers α_t and β_t and a homeomorphism h_t: $S^1\times[\alpha_t, \beta_t]\to\text{Int } S$ such that

(1) $-1<\alpha_t<t<\beta_t<1$,

(2) $h_t(y\times[\alpha_t, \beta_t])$ is a vertical line segment for each $y\in S^1$,

(3) $h_t(S^1\times r)$ lies in the horizontal plane $z=r$ for $r\in[\alpha_t, \beta_t]$,

(4) $T(h_t)$ lies in an λ_t-neighborhood of J_t, and

(5) $h_t(S^1\times t)$ is homeomorphically within δ_t of J_t.

A finite number of the tori $\{T(h_t)\}$ $(-1<t<1)$ suffice to cover $f_2(D)\cap S$. This finite collection of tori may be cut apart using horizontal planes $z=r$ with r rational to obtain a new finite collection $\mathcal{C}$ of disjoint tori which also cover $f_2(D)\cap S$. For each $T\in\mathcal{C}$ there exist a number t and rational numbers u and v such that $\alpha_t<u<v<\beta_t$ and $T=h_t(S^1\times[u, v])\cup A(h_t, u)\cup A(h_t, v)\cup S(u, v)$.

We next show that each component K of $f_2(D)\cap(T-S(u, v))$ is contained in the interior of an $\epsilon/3$-disk in $T-S(u, v)$. By (4) there is an open set in $\mathcal{C}_t$ which contains K; consequently, Diam $K<\delta_t$. Let $K'=\{x\in h_t(S^1\times t)\mid$ for some $y\in S^1$, $x=h_t(y\times t)$ and $K\cap h_t(y\times[u, v])\neq\emptyset\}$. It follows that Diam $K'<\delta_t$ so there is an $\epsilon/18$-arc B in $h_t(S^1\times t)$ that contains K'. If $M=\{x\in S^1\mid h_t(x\times t)\in B\}$ then the disk $E=\bigcup_{x\in M} h_t(x\times[u, v])$ is of diameter less than $\epsilon/9$ and contains $K\cap h_t(S^1\times[u, v])$. Since Diam $K\cap\text{Int } A(h_t, u)<\delta_t<\epsilon/9$, $\delta_t<\text{Diam } h_t(S^1\times u)$, and $A(h_t, u)$ lies in a horizontal plane, it follows that $K\cap\text{Int } A(h_t, u)$ does not separate J_u from $h_t(S^1\times u)$ on $A(h_t, u)$ and consequently lies in an $\epsilon/9$-subdisk E_u of $A(h_t, u)-J_u$. E_u may be chosen so that $E_u\cap E=\text{Bd } E_u\cap\text{Bd } E$ is an arc. Similarly, there is an $\epsilon/9$-subdisk E_v of $A(h_t, v)-J_v$ that contains $K\cap A(h_t, v)$ so that $E_v\cap E=\text{Bd } E_v\cap\text{Bd } E$ is an arc. It follows that $E_u\cup E\cup E_v$ is an $\epsilon/3$-subdisk of $T-S(u, v)$ that contains K.

Using techniques of the topology of E^2 it now follows that $(T-S(u, v))\cap f_2(D)$ is covered with a finite collection $\mathfrak{D}_T$ of disjoint $\epsilon/3$-disks in $T-S(u, v)$. The collection $\mathfrak{D}=\bigcup_{T\in\mathcal{C}}\mathfrak{D}_T$ is finite, disjoint, and such that $\bigcup_{F\in\mathfrak{D}} F$ separates $f_2(\text{Bd } D)$ from $f_2(D)\cap S$ on $f_2(D)$. It follows from the Tietze Extension Theorem, as indicated in [7, Lemma 1], that there is a map $g: D\to\text{Int } S$ such that $g\mid\text{Bd } D=f_2\mid\text{Bd } D$ and $g(D)\subset N(f_2(D), \epsilon/3)$.

From Steps I, II, and III we have $f\mid\text{Bd } D=f_1\mid\text{Bd } D=f_2\mid\text{Bd } D$ $=g\mid\text{Bd } D$ and $g(D)\subset N(f(D), \epsilon)$.

That S is tame from Int S now follows from Lemma 2, Lemma 3 and Bing's 1-ULC characterization of tame surfaces. Similar techniques are employed to show that S is tame from Ext S. Thus we have proved the following theorem suggested by Alexander [1].

THEOREM. *A 2-sphere S in E^3 is tame if each horizontal cross section of S is either a simple closed curve or a point.*

The author has recently learned that Norman Hosay has also given a proof of this theorem.

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