ON GENERATORS FOR VON NEUMANN ALGEBRAS[1]

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1. It has been conjectured that every von Neumann algebra on a separable Hilbert space has a single generator. The conjecture is true for type I algebras [3] and for hyperfinite algebras [7, Theorem 1]. T. Saitô [6] showed recently that for a certain class of von Neumann algebras, every algebra generated by two operators has a single generator. We show in §2 of this paper that every finitely generated algebra of the class has a single generator. In §3, we prove that every properly infinite von Neumann algebra on a separable Hilbert space is singly generated.

Throughout this paper, $\mathcal{H}$ will denote a separable complex Hilbert space. Operator always means bounded linear operator on a Hilbert space. $\mathcal{B}(\mathcal{H})$ is the set of bounded linear operators on $\mathcal{H}$. If $\mathcal{A}$ is a von Neumann algebra, then $\mathcal{A}'$ is the commutant of $\mathcal{A}$, and for $2 \leqq n \leqq \aleph_0$, $M_n(\mathcal{A})$ is the algebra of $n \times n$ matrices with entries in $\mathcal{A}$ which act boundedly on $\sum_{i=1}^{n} \oplus \mathcal{H}$. $\mathcal{R}(A, B, \cdots)$ denotes the von Neumann algebra generated by the family $\{A, B, \cdots\}$ of operators.

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2. If $\mathcal{A}$ is a von Neumann algebra, let $(*)$ be the property that $\mathcal{A}$ is $*$-isomorphic to $M_2(\mathcal{A})$. We will prove the following

THEOREM 1. *Let $\mathcal{A}$ be a von Neumann algebra which satisfies $(*)$ and suppose that $\mathcal{A}$ is finitely generated. Then $\mathcal{A}$ has a single generator.*

The following lemmas are needed in the proof of the theorem. These lemmas are generalizations of lemmas proved by T. Saitô in [6].

LEMMA 1. *Suppose a von Neumann algebra $\mathcal{A}$ is generated by n operators $A_1, A_2, \cdots, A_n$, $n \geqq 2$. Then $M_2(\mathcal{A})$ is generated by the $n+1$ operators*

$$\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} A_2 & 0 \\ 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} A_n & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}.$$

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The proof of this lemma is straightforward and is omitted.

LEMMA 2. *Suppose a von Neumann algebra* $\mathcal{A}$ *is generated by* $\{A_1, A_2, \cdots, A_n\}$, *where* A_1 *is normal and* $n \geqq 2$. *Then* $M_2(\mathcal{A})$ *is generated by* $n-1$ *operators.*

PROOF. We may suppose that $A_1, A_2, \cdots, A_n$ are invertible and are strict contractions. Let

$$B_i = \begin{pmatrix} A_i & 0 \\ 0 & 0 \end{pmatrix}$$

for $i = 1, \cdots, n-1$ and let

$$U = \begin{pmatrix} A_n & S_n \\ T_n & -A_n^* \end{pmatrix}$$

where $S_n = (I - A_n A_n^*)^{1/2}$ and $T_n = (I - A_n^* A_n)^{1/2}$. Then U is a unitary operator (cf. [2]), and B_1 is normal. Thus $\mathcal{R}(U)$ and $\mathcal{R}(B_1)$ are abelian von Neumann algebras, so $\mathcal{R}(B_1, U)$ has a single generator C by [3, top of p. 832]. Hence $\mathcal{R}(B_1, \cdots, B_{n-1}, U) = \mathcal{R}(C, B_2, \cdots, B_{n-1})$, so it remains to show that $M_2(\mathcal{A}) = \mathcal{R}(B_1, \cdots, B_{n-1}, U)$.

Let $\mathcal{R} = \mathcal{R}(B_1, \cdots, B_{n-1}, U)$. Then

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{R},$$

so that

$$X = \begin{pmatrix} A_n & 0 \\ T_n & 0 \end{pmatrix} = U \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{R}.$$

Then $M_2(\mathcal{R}(A_n)) = \mathcal{R}(X)$ by [4, Lemma 1]. Thus we have

$$\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} A_2 & 0 \\ 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} A_n & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \in \mathcal{R}$$

and $\mathcal{R} = M_2(\mathcal{A})$ by Lemma 1.

COROLLARY 1. *Let* $n \geqq 3$ *and suppose* $\mathcal{A}$ *is generated by the operators* $A_1, A_2, \cdots, A_n$ *where* A_1, A_2, A_3 *are normal. Then* $M_2(\mathcal{A})$ *is generated by* $n-2$ *operators.*

PROOF. $\mathcal{R}(A_2)$ and $\mathcal{R}(A_3)$ are abelian von Neumann algebras, so $\mathcal{R}(A_2, A_3)$ has a single generator B by [3, top of p. 832]. Then $\mathcal{A} = \mathcal{R}(A_1, B, A_4, \cdots, A_n)$ with A_1 normal, so by Lemma 2, $M_2(\mathcal{A})$ is generated by $n-2$ operators.

LEMMA 3. *Let $\mathfrak{A}$ be a von Neumann algebra generated by $\{A_1, A_2, \cdots, A_n\}$, $n \geqq 2$. Then $M_2(\mathfrak{A})$ is generated by $n+1$ unitary operators.*

PROOF. We suppose $A_1, A_2, \cdots, A_n$ are invertible strict contractions. Let

$$W = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad U_i = \begin{pmatrix} A_i & S_i \\ T_i & -A_i^* \end{pmatrix}, \qquad i = 1, \cdots, n,$$

where $S_i = (I - A_i A_i^*)^{1/2}$ and $T_i = (I - A_i^* A_i)^{1/2}$. The U_i are unitary and W is a symmetry. Then

$$\begin{pmatrix} A_1 & 0 \\ T_1 & 0 \end{pmatrix} = U_1 \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{R}(U_1, W)$$

since

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = \tfrac{1}{2}(I + W).$$

Thus we see that $\mathfrak{R}(U_1, W) = M_2(\mathfrak{R}(A_1))$ by [4, Lemma 1]. Therefore

$$\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \in M_2(\mathfrak{R}(A_1)) = \mathfrak{R}(U_1, W)$$

and we have

$$\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} A_2 & 0 \\ 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} A_n & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$$
$$\in \mathfrak{R}(U_1, U_2, \cdots, U_n, W).$$

Hence $\mathfrak{R}(U_1, U_2, \cdots, U_n, W) = M_2(\mathfrak{A})$ by Lemma 1.

PROOF OF THEOREM 1. Suppose $\mathfrak{A}$ is generated by n operators, $n \geqq 2$. Then $\mathfrak{A}$ is generated by $n+1$ unitary operators by Lemma 3. Hence $\mathfrak{A}$ is generated by $(n+1) - 2 = n - 1$ operators by Corollary 1. It follows that $\mathfrak{A}$ has a single generator.

REMARK 1. In the above proof, if we do not assume that $\mathfrak{A}$ satisfies (*), then the argument shows that if $\mathfrak{A}$ is generated by $n \geqq 2$ operators, $M_4(\mathfrak{A})$ is generated by $n-1$ operators. Hence

COROLLARY 2. *If $\mathfrak{A}$ is a von Neumann algebra generated by $n \geqq 2$ operators, then $M_{4^{n-1}}(\mathfrak{A})$ is singly generated.*

REMARK 2. The question of which von Neumann algebras satisfy (*) is only partially answered. It is known that properly infinite algebras and type II_1 hyperfinite factors satisfy (*).

3. In this section we prove the following

THEOREM 2. *If $\mathfrak{A}$ is a properly infinite von Neumann algebra (i.e., $\mathfrak{A}$ contains no finite central projections) on a separable Hilbert space, then $\mathfrak{A}$ has a single generator.*

We prove this theorem by means of

LEMMA 4. *Suppose a von Neumann algebra $\mathfrak{A}$ is generated by n opera tors, $2 \leqq n \leqq \aleph_0$. Then $M_n(\mathfrak{A})$ is doubly generated.*

PROOF. Let $\mathfrak{A} = \mathfrak{R}(\{A_k\}_{k=1}^n)$, $1 \leqq n \leqq \aleph_0$. We can suppose that $\|A_k\| \leqq 1$ for all k. Define $A \in M_n(\mathfrak{A})$ by

$$A = \begin{pmatrix} A_1 & & & \\ & A_2 & & 0 \\ & & A_3 & \\ 0 & & & \ddots \end{pmatrix}$$

If I is the identity operator on $\mathfrak{H}$, then clearly $M_n(\mathbf{C}I) \subset M_n(\mathfrak{A})$, where $\mathbf{C}$ denotes the complex numbers. $M_n(\mathbf{C}I)$ is a type I factor, so (e.g. by [3]), there is an operator $B \in M_n(\mathbf{C}I)$ with $\mathfrak{R}(B) = M_n(\mathbf{C}I)$. We assert that $\mathfrak{R}(A, B) = M_n(\mathfrak{A})$. Suppose $C \in \mathfrak{R}(A, B)'$ with C self-adjoint. Then $C \in \mathfrak{R}(B)' = M_n(\mathbf{C}I)'$, so

$$C = \begin{pmatrix} D & & & \\ & D & & 0 \\ & & D & \\ 0 & & & \ddots \end{pmatrix}$$

for some selfadjoint $D \in \mathfrak{B}(\mathfrak{H})$. C also commutes with A, so D commutes with A_k for all k. Therefore $D \in \mathfrak{A}'$. Thus

$$\mathfrak{R}(A, B)' = \left\{ \begin{pmatrix} D & & & \\ & D & & 0 \\ & & D & \\ 0 & & & \ddots \end{pmatrix} : D \in \mathfrak{A}' \right\}.$$

Then clearly $\mathfrak{R}(A, B) = \mathfrak{R}(A, B)'' = M_n(\mathfrak{A})$.

PROOF OF THEOREM 2. Since $\mathfrak{A}$ is properly infinite, we know that $\mathfrak{A}$ is $*$-isomorphic to $M_n(\mathfrak{A})$ for $1 \leqq n \leqq \aleph_0$ (cf. [5, p. 458]). Choose an at most countable set $\{A_k\}_{k=1}^n$ of operators which generates $\mathfrak{A}$ (cf.

[1, p. 33]). By Lemma 4, $\mathfrak{A}$ is doubly generated, so by Theorem 1, $\mathfrak{A}$ has a single generator.

The following is a corollary of Lemma 4.

COROLLARY 3. *If a von Neumann algebra $\mathfrak{A}$ is generated by n operators, $1 \leqq n < \infty$, then $M_{4n}(\mathfrak{A})$ has a single generator.*

PROOF. $M_n(\mathfrak{A})$ is doubly generated by Lemma 4, so $M_{4n}(\mathfrak{A})$ is singly generated by Corollary 2.

REFERENCES

1. J. Dixmier, *Les algebres d'operateurs dans l'espace Hilbertien*, Gauthier-Villars, Paris, 1957.

2. P. R. Halmos, *Normal dilations and extensions of operators*, Summa Brasil. Math. **2** (1950), 125–134.

3. C. Pearcy, *W*-algebras with a single generator*, Proc. Amer. Math. Soc. **13** (1962), 831–832.

4. ———, *On certain von Neumann algebras which are generated by partial isometries*, Proc. Amer. Math. Soc. **15** (1964), 393–395.

5. C. Pearcy and D. Topping, *Sums of small numbers of idempotents*, Michigan Math. J. **14** (1967), 453–465.

6. T. Saitô, *A remark on generators of von Neumann algebras*, Michigan Math. J. **16** (1969), 1–4.

7. N. Suzuki and T. Saitô, *On the operators which generate continuous von Neumann algebras*, Tôhoku Math. J. **15** (1963), 277–280.

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