# FOURIER SERIES WITH POSITIVE COEFFICIENTS

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I shall state a number of results on sine or cosine series with nonnegative coefficients; proofs of these and some related theorems will appear elsewhere.

The following theorems are known.

A [1], [8]. *If* $\lambda_n \downarrow 0$, $\phi(x) = \sum \lambda_n \cos nx$, *and* $0<\gamma<1$, *then* $\sum n^{\gamma-1}\lambda_n < \infty$ *if and only if* $x^{-\gamma}\phi(x) \in L$.

A′ [6]. *If* $\lambda_n$ *are the Fourier coefficients of* $\phi$, $\lambda_n \geqq 0$, *and* $1<\gamma<3$, *then* $\sum n^{\gamma-1}\lambda_n < \infty$ *if and only if* $x^{-\gamma}[\phi(x)-\phi(0)] \in L$.

B [2], [3]. *If* $\lambda_n \downarrow 0$, $\phi(x) = \sum \lambda_n \cos nx$, $1<p<\infty$, *and* $(1-p)/p < \gamma < 1/p$, *then* $x^{-\gamma}\phi(x) \in L^p$ *if and only if* $\sum n^{p+p\gamma-2}\lambda_n^p < \infty$.

C [7]. *If* $\lambda_n \downarrow 0$, $\phi(x) = \sum \lambda_n \cos nx$, *and* $0<\gamma<1$, *then* $\phi(x) \in \mathrm{Lip}\,\gamma$ *if and only if* $\lambda_n = O(n^{-\gamma-1})$.

There are similar theorems for sine series.

The following theorems generalize A and C (with different necessary and sufficient conditions), to series with nonnegative coefficients, and give a result that is related to B as A′ is related to A.

THEOREM 1. *If* $\lambda_n \geqq 0$, $\lambda_n$ *are the Fourier sine or cosine coefficients of* $\phi$ *and* $0<\gamma<1$, *then*

$$\sum n^{\gamma-1}\lambda_n < \infty \tag{1}$$

*if and only if*

$$\int_{a+}^{\pi} (x-a)^{-\gamma}\phi(x)dx \text{ converges}, \quad 0 \leqq a < \pi. \tag{2}$$

More precisely, (1) is necessary for (2) with $a=0$ and sufficient for (2) for all $a$—an illustration of the principle that a Fourier series with nonnegative coefficients tends to behave as well at all points as it does at 0. (The case $a=0$ is a special case of a more general result of Edmonds [4, p. 235].) Theorem A′ can be generalized in the same way if $1<\gamma<2$.

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THEOREM 2. *If* $\lambda_n \geqq 0$ *and* $\lambda_n$ *are the Fourier sine or cosine coefficients of* $\phi$, *and* $1/p<\gamma<(p+1)/p$, *then*

$$|x - a|^{-\gamma} \quad |\phi(x) - \phi(a)| \in L^p, \quad 0 \leqq a < \pi, \tag{3}$$

*if and only if*

$$\sum_{n=1}^{\infty} n^{p\gamma-2}\left(\sum_{k=n}^{\infty} \lambda_k\right)^p < \infty. \tag{4}$$

More precisely, (4) is necessary for (3) if $a=0$ and sufficient for (3) for all $a$. Theorem B can be obtained as a corollary.

THEOREM 3. *If* $\lambda_n \geqq 0$, $\lambda_n$ *are the Fourier sine or cosine coefficients of* $\phi$, *and* $0<\gamma<1$, *then* $\phi \in \mathrm{Lip}\,\gamma$ *if and only if*

$$\sum_{k=n}^{\infty} \lambda_k = O(n^{-\gamma}). \tag{5}$$

When $\lambda_k \downarrow 0$, (5) is equivalent to $\lambda_n = O(n^{-1-\gamma})$, so Theorem C is a corollary. Theorem 3 is formally the limiting case $p=\infty$ of Theorem 2.

Theorem 3 fails when $\gamma=1$. There are a number of substitutes, among them the following result, in which $\Lambda_*$ and $\lambda_*$ denote the classes of continuous functions $\phi$ such that $\phi(x+h)+\phi(x-h)-2\phi(x) = O(h)$ or $o(h)$, uniformly in $x$ [**10**, p. 43].

THEOREM 4. *If* $\lambda_n \geqq 0$ *and* $\lambda_n$ *are the Fourier cosine coefficients of* $\phi$, *then* (5) *with* $\gamma=1$ *is a necessary and sufficient condition for either* $f(x)-f(0)=O(x)$ *or* $f\in\Lambda_*$;

$$\sum_{k=n}^{\infty} \lambda_k = O(n^{-1}) \tag{6}$$

*is necessary and sufficient for either* $f(x)-f(0)=o(x)$ *or* $f \in \lambda_*$; *if* (6) *holds, then* $f'(x)$ *exists* [$f'$ *is continuous*] *if and only if* $\sum k\lambda_k \sin kx$ *converges* [*converges uniformly*].

Paley (see [**5**, p. 72]; [**9**]) showed that if the sine series of a continuous function has nonnegative coefficients then the series converges uniformly. As a corollary of Theorem 4 we have a localization of this.

THEOREM 5. *If* $\phi$ *has nonnegative sine coefficients* $\lambda_n$ *and*

$$\sum_{n}^{\infty} k^{-1}\lambda_k = O(1/n) \tag{7}$$

*then* $\sum \lambda_k \sin kx$ *converges (for any particular* $x$*) if and only if* $\phi$ *is the derivative of its integral at* $x$.

In fact, if $\phi$ is continuous, $\int \phi \in \lambda_*$ and so (7) holds.

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