ISOMORPHIC COMPLEXES. II

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In a preceding note [2] we showed that if K and L are n-complexes, then K and L are isomorphic iff the 1-sections of the first derived complexes of K and L are isomorphic. Since topological equivalence does not imply combinatorial equivalence for complexes this result fails to hold if the 1-sections are only required to be homeomorphic. However, for a large class of complexes we will show that this theorem is true under the weaker condition.

Throughout, s_p will denote a (rectilinear) p-simplex with vertices $a^0, a^1, \cdots, a^p$; K will denote a finite geometric simplicial complex with n-section K^n and first derived complex K'. For more details see [1, §1.2].

We first recall a definition and two theorems from [2].

DEFINITION 1. An n-complex K is *full* provided, for any subcomplex L of K which is isomorphic to s_p^1, $2 \leqq p \leqq n$, L^0 spans a p-simplex of K.

THEOREM 1. *If K and L are full n-complexes, then K and L are isomorphic iff K^1 and L^1 are isomorphic.*

THEOREM 2. *If K and L are n-complexes, then K and L are isomorphic iff $(K')^1$ and $(L')^1$ are isomorphic.*

DEFINITION 2. A complex K is said to be *taut* provided, K^1 has no vertex of order 2.

DEFINITION 3. A complex K is said to be *trim* if it is full and taut.

In each of the next three theorems we need only prove one implication for the equivalence since isomorphic complexes have homeomorphic realizations.

THEOREM 3. *If K and L are taut 1-complexes, then K and L are isomorphic iff $|K|$ and $|L|$ are homeomorphic.*

PROOF. Let $\phi: |K| \rightarrow |L|$ be a homeomorphism of $|K|$ onto $|L|$. If a is a vertex of K, then the order of $\phi(a)$ is not two since order is a topological property. So $\phi(a)$ is a vertex of L. Hence L has at least as many vertices as K. Similarly, using ϕ^{-1} instead of ϕ we obtain that K has at least as many vertices as L. So K and L have the same number of vertices. Therefore, $v: K \rightarrow L$ defined by

$$v(a) = \phi(a), \qquad a \in K^0$$

is a vertex transformation of K to L taking K^0 onto L^0 in a 1-1 fashion.

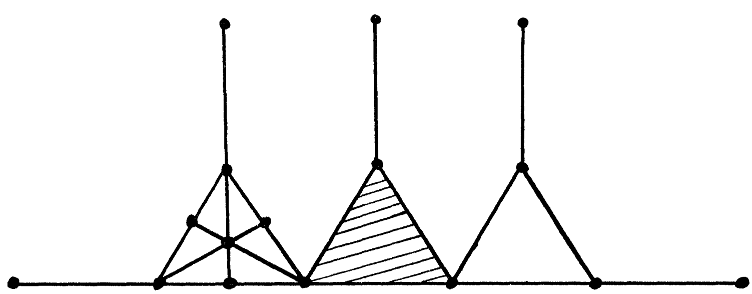
We will now show that v is admissible. If a^0, a^1 span a 1-simplex of K, then $v(a^0)$, $v(a^1)$ span a 1-simplex of L since $v(a^0), v(a^1) \in L^0$ are the end points of the arc $\phi[|a^0a^1|]$ which contains no other vertices of L. So v is admissible. A similar argument shows v^{-1} is also an admissible vertex transformation. Hence v induces an isomorphism of K onto L.

THEOREM 4. *If K and L are trim n-complexes, then K and L are isomorphic iff $|K^1|$ and $|L^1|$ are homeomorphic.*

PROOF. Suppose $|K^1|$ and $|L^1|$ are homeomorphic. Then since they are taut, we have K^1 and L^1 are isomorphic by Theorem 3. Since K and L are full, Theorem 1 applies and so K and L are isomorphic.

THEOREM 5. *If K' and L' are taut n-complexes, then K and L are isomorphic iff $|(K')^1|$ and $|(L')^1|$ are homeomorphic.*

PROOF. Suppose $|(K')^1|$ and $|(L')^1|$ are homeomorphic. Then since they are taut we have $(K')^1$ and $(L')^1$ are isomorphic by Theorem 3. So Theorem 2 applies and we have that K and L are isomorphic.



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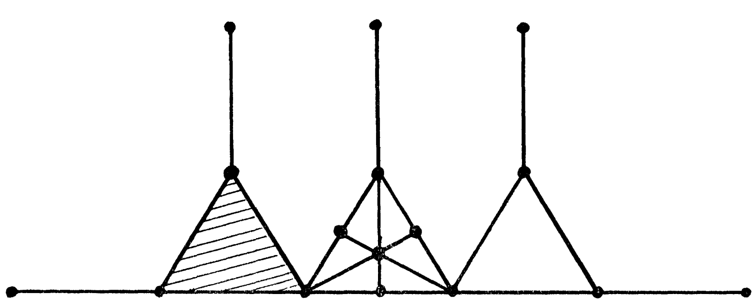

(L)

EXAMPLE 1. Let $K=(s_2^1)'$ and $L=K'$. Then K and L are nonisomorphic full 1-complexes and $|K^1|$ and $|L^1|$ are homeomorphic. This shows the need for requiring tautness in Theorems 4 and 5.

EXAMPLE 2. That tautness of K and L is not a strong enough requirement in Theorem 5 is shown by the preceding example of two taut nonisomorphic 2-complexes with $|(K')^1|$ and $|(L')^1|$ being homeomorphic.

REFERENCES

1. P. J. Hilton and S. Wiley, *Homology theory*, Cambridge Univ. Press, New York, 1960.

2. J. Segal, *Isomorphic complexes*, Bull. Amer. Math. Soc. **71** (1965), 571–572.

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