A CONJECTURE OF J. NAGATA ON DIMENSION AND METRIZATION[1]

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THEOREM 1. *A metrizable space X is of dimension $\leqq n$ if and only if X admits a metric compatible with the topology which satisfies the condition* (n): *For any $n+3$ points $x, y_1, \cdots, y_{n+2}$ of X there exist distinct indices i, j such that $d(y_i, y_j) \leqq d(x, y_j)$.*

In this paper we outline briefly a proof of Theorem 1, which was conjectured by J. Nagata [1].

By dimension we shall always mean covering dimension. A family of subsets of X is discrete if each point of X has a neighborhood which meets at most one member of the family. For a subset A of X and a family $\mathcal{C}$ of subsets of X, let $S(A, \mathcal{C})$ denote the union of A and all those $C \in \mathcal{C}$ such that $C \cap A \neq \emptyset$. For each integer $n \geqq 0$, let

$$S^n(A, \mathcal{C}) = \begin{cases} A & \text{if } n = 0, \\ S(S^{n-1}(A, \mathcal{C}), \mathcal{C}) & \text{if } n > 0; \end{cases}$$

$$[\mathcal{C}]^n = \{S^n(C, \mathcal{C}) : C \in \mathcal{C}\}.$$

Let X be a metrizable space of dimension $\leqq n$. For each positive integer j there exist $n+1$ discrete families of open sets, $\mathcal{U}_j^1, \mathcal{U}_j^2, \cdots, \mathcal{U}_j^{n+1}$ such that if $\mathcal{U}_j = \bigcup_{i=1}^{n+1} \mathcal{U}_j^i$, then:

(1) each $\mathcal{U}_j$ covers X;

(2) for each $x \in X$, $\{S(x, \mathcal{U}_j) : j = 1, 2, \cdots\}$ is a neighborhood base at x;

(3) $[\mathcal{U}_{j+1}]^{31}$ refines $\mathcal{U}_j$ for each j;

(4) if $j < k$ and $1 \leqq i \leqq n+1$, each member of $[\mathcal{U}_k]^{31}$ meets at most one member of $\mathcal{U}_j^i$.

The $\mathcal{U}_j^i$ are defined inductively on j. Their construction relies on a new characterization of dimension [2].

THEOREM 2. *A metrizable space X is of dimension $\leqq n$ if and only if for each open cover $\mathcal{C}$ of X there exist $n+1$ discrete families of open sets, $\mathcal{U}^1, \mathcal{U}^2, \cdots, \mathcal{U}^{n+1}$ such that $\bigcup_{i=1}^{n+1} \mathcal{U}^i$ is a cover of X which refines $\mathcal{C}$.*

PROOF OF THEOREM 1. Let R^* denote the set of dyadic rationals in the open interval $(0, 1)$. For each $m \in R^*$ there exist $n+1$ discrete

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families of open sets $\mathcal{S}_m^1, \mathcal{S}_m^2, \cdots, \mathcal{S}_m^{n+1}$ such that if $\mathcal{S}_m = \bigcup_{i=1}^{n+1} \mathcal{S}_m^i$ then:

(1′) each $\mathcal{S}_m$ covers X;

(2′) for each $x \in X$, $\{S(x, \mathcal{S}_m): m \in R^*\}$ is a neighborhood base at x;

(3′) if $m < p \in R^*$, then $\mathcal{S}_m$ refines $\mathcal{S}_p$;

(4′) if $m, p \in R^*$, and j is a positive integer such that $2^{-j} \leqq m < p \leqq 2^{-(j-1)}$, then $\mathcal{S}_m^i$ refines $\mathcal{S}_p^i$ for each $1 \leqq i \leqq n+1$;

(5′) if $m < p \in R^*$, $1 \leqq i \leqq n+1$ and $U \in \mathcal{S}_m^i$, $V \in \mathcal{S}_p^i$, then either $U \subset V$ or $U \cap V = \varnothing$;

(6′) if $m, p \in R^*$, $m+p < 1$, and if $U \in \mathcal{S}_m$, $V \in \mathcal{S}_p$ are such that $U \cap V \neq \varnothing$, then there exists $W \in \mathcal{S}_{m+p}$ such that $U \cup V \subset W$.

The $\mathcal{S}_m^i$ are constructed from the $\mathcal{U}_j^i$ as follows: Let

$$*\mathcal{U}_j^i = \{S^{31}(U, \mathcal{U}_{j+1}): U \in \mathcal{U}_j^i\}; \qquad *\mathcal{U}_j = \bigcup_{i=1}^{n+1} *\mathcal{U}_j^i.$$

For $A \subset X$, $1 \leqq i \leqq n+1$, $j \geqq 1$ and $k \geqq 0$, let

$$T^k(A, i, j) = \begin{cases} S(A, *\mathcal{U}_j^i) & \text{if } k = 0, \\ S(T^{k-1}(A, i, j), *\mathcal{U}_{j+k}^i) & \text{if } k > 0; \end{cases}$$

$$T(A, i, j) = \bigcup_{k=0}^{\infty} T^k(A, i, j).$$

For $m \in R^*$ of the form $m = \sum_{k=1}^{t} 2^{-m_k}$, where $1 \leqq m_1 < m_2 < \cdots < m_t$, let $\bar{m} = \sum_{k=1}^{t-1} 2^{-m_k}$. Further, for $A \subset X$ and $1 \leqq i \leqq n+1$, let

$$i_A m = \begin{cases} T(A, i, m_1 + 1) & \text{if } t = 1, \\ T(S^3(i_A \bar{m}, *\mathcal{U}_{m_t}), i, m_t) & \text{if } t > 1; \end{cases}$$

$$\mathcal{S}_m^i = \{i_U m: U \in \mathcal{U}_{m_1}^i\}.$$

This complicated construction is necessary to achieve condition (5′). A simpler construction in which (5′) does not hold was used by Nagata [1] to obtain a weaker result.

Define a non-negative real-valued function d on $X \times X$ by

$$d(x, y) = \begin{cases} 1 & \text{if } y \notin S(x, \mathcal{S}_m) \text{ for any } m \in R^*, \\ \inf\{m \in R^*: y \in S(x, \mathcal{S}_m)\} & \text{if } y \in S(x, \mathcal{S}_m) \text{ for some } m \in R^*. \end{cases}$$

d is clearly symmetric. It follows from (3′) that $d(x, y) = 0$ only if $x = y$. The triangular inequality follows from (6′). Thus d is a metric on X. By (1′) and (2′) d is compatible with the topology of X. If $x, y_1, \cdots, y_{n+2}$ are points of X, then by (4′) and (5′) there exist indices i, j, $i \neq j$, such that $d(y_i, y_j) \leqq d(x, y_j)$.

Conversely, suppose that X is a metric space with metric d satisfying condition (n). Let $\mathcal{C}$ be an open cover of X. It is easily shown by use of Zorn's Lemma that if $A \subset X$ and $\epsilon > 0$, there is a subset B of X which is maximal (under inclusion) with respect to the properties:

(i) $B \cap A = \emptyset$;

(ii) $\{S_\epsilon(x) : x \in B\}$ refines $\mathcal{C}$; $(S_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\})$

(iii) if $x \neq y \in B$, then $d(x, y) \geqq \epsilon$.

Hence we may construct subsets A_i of X for $i = 1, 2, \cdots$, inductively on i, which are maximal with respect to the properties:

(1) $A_i \cap \{y \in X : \text{for some } 1 \leqq j < i \text{ and } x \in A_j, d(x, y) < 2^{-i}\} = \emptyset$;

(2) $\{S_{2^{-i}}(x) : x \in A_i\}$ refines $\mathcal{C}$;

(3) if $x \neq y \in A_i$, then $d(x, y) \geqq 2^{-i}$.

Let $\mathcal{U} = \{S_{2^{-i}}(x) : x \in A_i; i = 1, 2, \cdots\}$. The maximality of the A_i insures that $\mathcal{U}$ covers X. Conditions (1) and (3) and the condition on the metric insure that $\mathcal{U}$ is of order $\leqq n+1$. Thus the proof is complete.

References

1. J. Nagata, *General topology and its relation to modern analysis and algebra*, Academic Press, New York; pp. 282–285.

2. P. A. Ostrand, *Dimension of metric spaces and Hilbert's Problem* 13, Bull. Amer. Math. Soc. 71 (1964), 619–622.

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