JORDAN ALGEBRAS OF SELF-ADJOINT OPERATORS[1]

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1. **Introduction.** The purpose of this note is to announce a *real* noncommutative (more precisely, nonassociative) generalization and counterpart to the theory of von Neumann algebras. The algebras in question are weakly closed Jordan algebras of self-adjoint (s.a.) operators, to be referred to below as JW-*algebras.*

The results obtained are remarkably parallel to the global von Neumann theory. Our principal contributions are the development of a theory of relative dimension, culminating in the Comparison Theorem (from which a variety of structural information is obtained) together with an example of a new factor phenomenon not occurring in the von Neumann theory. Details and proofs will be published in the Memoirs of the Society.

2. **Quadratic ideals and annihilators.** Let A be a JW-algebra and M any subset of A. The *annihilator* of M is the set $M^{\perp}=\{a\in A: ab=0$ for all $b\in M\}$ (ab denotes the ordinary operator product).

A *quadratic ideal* is a linear subspace I of A with $aba\in I$ whenever $a\in I$ and $b\in A$ (note that $aba=2a\circ(a\circ b)-a^2\circ b$, where $a\circ b=\frac{1}{2}(ab+ba)$). The *center* of A is the set $Z=\{z\in A: za=az$ for all $a\in A\}$.

THEOREM 1. *The annihilators in a* JW-*algebra A are precisely the weakly closed quadratic ideals, and are of the form $eAe=\{eae: a\in A\}$, where e is a projection in A. The projections form a complete orthomodular lattice (so A has a largest projection which we assume is the identity operator* 1). *The annihilator of a Jordan ideal has the form eAe with e central. For a projection $e\in A$, eAe is a Jordan ideal if and only if e is central. The annihilator of a central subset is a direct summand.*

As usual, we define the *central cover* $C(a)$ of $a\in A$ to be the smallest central projection e with $ea=a$. We call a *faithful* if $C(a)=1$.

COROLLARY 1. *The central cover $C(a)$ exists and is the unique central projection e for which $(a)^{\perp}=\{e\}^{\perp}$, where (a) is the principal Jordan ideal generated by a.*

[1] Most of this work was completed while the author held a NATO Postdoctoral Fellowship.

A projection $e \in A$ is termed *abelian* if the operators in eAe commute (i.e. if eAe, is associative as a Jordan algebra). A JW-algebra A is *type* I if it possesses a faithful abelian projection.

THEOREM 2. *In any* JW-*algebra* A, *there is a largest type* I *summand.*

3. **Dimension theory.** We define two projections e and f in a JW-algebra A to be *equivalent* (written $e \sim f$) if there is a finite sequence $s_1, \cdots, s_n$ of symmetries ($=$s.a. unitaries) in A with $u^*eu = f$, where $u = s_1 \cdots s_n$. We say that e and f are *exchanged by a symmetry* s if $ses = f$.

REMARKS. Equivalence need not be finitely additive except when the projection lattice is modular (and hence a continuous geometry), and then equivalence is completely additive. If A is the s.a. part of a von Neumann algebra, this equivalence relation is the same as perspectivity and unitary equivalence (this is a recent result of P. A. Fillmore).

Projections e and f in A are *related* if they have equivalent nonzero subprojections in A.

THEOREM 3. *Equivalence is the same as projectivity. For a projection* $e \in A$, $C(e) = \mathrm{LUB}\{f : f \precsim e\}$ *and* $C(e) \perp C(f)$ *if and only if* e *and* f *are unrelated. Also* $C(e) = \mathrm{LUB}\{sfs : f \leqq e\}$ *where* s *ranges over all symmetries in* A. *Two related projections in* A *have nonzero subprojections in* A *which are exchanged by a symmetry in* A.

The next result more than justifies our notion of equivalence. Even for von Neumann algebras, this is a strengthening of the familiar technique.

THEOREM 4 (THE COMPARISON THEOREM). *Given any two projections* e *and* f *in a* JW-*algebra* A, *there is a central projection* $h \in A$ *and a symmetry* $s \in A$ *with* $s(eh)s \leqq fh$ *and* $s(f(1-h))s \leqq e(1-h)$.

We call a projection e *modular* if eAe has a modular projection lattice (A is called *modular* if 1 is).

THEOREM 5. *The following "global" conditions on a* JW-*algebra* A *are equivalent*:

(1) *Each projection* $e \in A$ *satisfies*: $e \sim f \in A$ *and* $f \leqq e$ *imply* $f = e$ *("finiteness").*

(2) *A has a modular projection lattice.*

(3) *Every orthogonal family of equivalent projections in* A *is finite.*

If e *and* f *are modular, so is* $e \cup f$. *On the set of modular projections, perspectivity is transitive. If* e *is modular and* $f \sim e$ *with* $f \leqq e$, *then* $f = e$.

If e and f are modular in A, then $e \precsim f$ and $f \precsim e$ imply that e and f are exchanged by a symmetry in A.

A JW-algebra is *locally modular* if it has a faithful modular projection. We say that A is *properly nonmodular* if A has no central modular projections except zero, and *purely nonmodular* if A contains no modular projections except zero.

THEOREM 6. *Any* JW-*algebra A decomposes uniquely into five summands as follows*:

(1) *Type* I *modular* (*"finite, discrete"*).
(2) *Type* I *properly nonmodular* (*"properly infinite, discrete"*).
(3) *Type* II *modular* (*"finite, continuous"*).
(4) *Type* II *properly nonmodular* (*"properly infinite, continuous"*).
(5) *Type* III *purely nonmodular* (*"purely infinite"*).

(*Types* II *and* III *are summands of the non-type* I *portion determined in Theorem* 2.)

REMARK. If A is the s.a. part of a von Neumann algebra, this decomposition into types coincides with the classical one, even though the equivalence relations differ in the "infinite" cases.

4. Structure. The dimension theory just mentioned can be applied to yield structural data.

THEOREM 7. *For any projection $e \in A$, the center of eAe is Ze, where Z is the center of A. Hence if A is a factor (Z = the reals), then eAe is also.*

We say that a JW-algebra is *homogeneous* if there is an orthogonal family $\{e_i\}$ of abelian projections, any two of which are exchanged by a symmetry in A, such that LUB $e_i = 1$.

THEOREM 8. *Two abelian projections in a* JW-*algebra A with the same central cover are exchanged by a symmetry in A. Any type* I JW-*algebra is a product of homogeneous algebras. The "spectral multiplicity" of a homogeneous* JW-*algebra is unique.*

A JW-algebra can also be separated into atomic and nonatomic portions. Regarding the "continuous" summand of a JW-algebra, we have

THEOREM 9. *If A is a* JW-*algebra having no type* I *portion, then any projection in A can be split into two orthogonal halves which are exchanged by a symmetry in A.*

We call A *simple* if it has no nontrivial Jordan ideals, and *strongly semisimple* if the intersection of its maximal Jordan ideals is zero.

THEOREM 10. *A modular* JW-*factor is simple and any modular* JW-*algebra is strongly semisimple. All* JW-*algebras are weakly central. A* JW-*algebra and its center have homeomorphic maximal (Jordan) ideal spaces. The norm-closed Jordan ideals of any* JW-*algebra are in a* 1-1 *correspondence with the p-ideals of its projection lattice.*

5. **Dimension and trace.** We have employed some recent results of Arlan Ramsay to obtain

THEOREM 11. *Any locally modular* JW-*algebra A possesses a canonical dimension function d on its projection lattice L, taking values in the continuous extended real-valued functions on the Stone space of the center of L and satisfying:*

(1) *$d(e)=0$ if and only if $e=0$.*

(2) *d is completely additive.*

(3) *$d(ses)=d(e)$ for all symmetries $s\in A$.*

(4) *The projection e is modular if and only if $d(e)$ is finite except on a nowhere dense set.*

If we define $e\approx f$ to mean $d(e)=d(f)$, then $(L, \approx)$ is a dimension lattice satisfying axioms (A), (B), (C), (D′) and (M) of Loomis (*The lattice theoretic background of the dimension theory of operator algebras*, Mem. Amer. Math. Soc. No. 18 (1955), 36 pp.) such that central projections are $\approx$-invariant. If, in addition, L satisfies:

(*) *Each orthogonal family of modular projections having a common central cover is countable,*

then $(L, \approx)$ is the unique dimension lattice structure agreeing with $\sim$ (=perspectivity, by Theorem 5) on the p-ideal of modular projections.

REMARKS. Any JW-algebra acting on a separable Hilbert space satisfies (*). A locally modular JW-factor satisfies (*) if and only if it is countably decomposable. Any direct product of locally modular JW-algebras satisfying (*) also satisfies (*).

The type III summand of a JW-algebra can be given a type III dimension lattice structure $(L, \approx)$ making central projections $\approx$-invariant by defining $e\approx f$ to mean that e and f have the same central cover.

By a *center-valued trace* we shall mean a normalized positive linear map ϕ of a JW-algebra A onto its center Z such that $\phi(za)=z\phi(a)$

and $\phi(sas)=\phi(a)$ for $a\in A$, $z\in Z$ and each symmetry $s\in A$.

Let G be the group of all finite products of symmetries from A. For $a\in A$, let K_a be the norm-closed convex hull of the orbit of a under G acting by $(u, a)\rightarrow uau^*$, where $u\in G$ and $a\in A$.

THEOREM 12. *For each $a\in A$, $K_a\cap Z$ is nonempty* (THE APPROXIMATION THEOREM). *A* JW-*algebra is modular if and only if it possesses a completely additive faithful center-valued trace. For such an algebra, the trace is unique.*

6. **A new kind of factor.** The JW-factor referred to is type I modular ("discrete, finite class") but infinite-dimensional as a real linear space. We outline the construction of this factor below.

Let $\{s_i\}$ be a (countably) infinite sequence of anticommuting symmetries (satisfying the Pauli "spin relations" $s_is_k+s_ks_i=2\delta_{ik}$) in a type II_1 hyperfinite von Neumann factor acting on a separable Hilbert space. We take A to be the closure, in the weak operator topology, of the real linear space spanned by the identity operator and the sequence $\{s_i\}$. This is easily seen to be a JW-algebra.

The difficulty arises in showing that A is actually a factor. This is done geometrically by showing that A (partially ordered by its cone of positive semi-definite operators) is an *antilattice* in the sense that two operators in A have a greatest lower bound there only in the case where they are comparable in the ordering.

Our factor A inherits a trace from the von Neumann factor enveloping it, and, on the projection lattice of A, the trace takes three values: 0, $\frac{1}{2}$ and 1. Each projection ($\neq 0, 1$) in A is both maximal and minimal (and therefore abelian). The maximal associative Jordan subalgebras of A are just the planes passing through 0 and 1. Any two projections ($\neq 0, 1$) are exchanged by a symmetry in A, and A is homogeneous of "spectral multiplicity" two.

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