BOOK REVIEWS

Flows on homogeneous spaces. By L. Auslander, L. Green, and F. Hahn, with the assistance of L. Markus and W. Massey and an Appendix by L. Greenberg. Annals of Mathematics Studies Number 53. Princeton University Press, Princeton, N. J., 1963. 7+107 pp. $2.75.

This highly significant book should be pleasing to a wide variety of mathematical tastes. It is a study by diverse methods of the topological and measure-theoretic behavior of flows associated with Lie groups. The contents should probably be described neither as general theorems nor special examples, but either as relatively special theorems or relatively general examples.

A one-parameter subgroup of a Lie group G acts on a homogeneous space G/D of G by left multiplication to produce a "coset" flow, a flow being a transformation group determined by the real line acting upon a phase space which in this case is G/D. This collection of closely related papers forms an extended investigation of coset flows, where G is usually assumed to be nilpotent or, more generally, solvable, with respect to properties centrally located in topological dynamics and ergodic theory. The techniques which the authors use range over Lie groups, Lie algebras, general topology, homology theory, fiber bundles, differential geometry, measure theory, functional analysis, spectral theory, and infinite-dimensional group representations, with a bonus appearing in connection with diophantine approximation. An impressive exhibition of breadth and depth in mathematical knowledge and skill, the book is a record of some of the research work done under the auspices of the NSF-sponsored conference "Analysis in the Large" at Yale University during the academic year 1960–61.

Chapters I and II are largely summaries of known theories and are intended to smooth the reader's path into the complicated considerations of the ensuing parts of the book. The chapters and appendices after Chapter II are research papers which contain many new and interesting results. The contents of these one hundred closely packed pages can only be cursorily and very inadequately indicated here by at most a few sentences for each chapter and appendix. Any particular theorems quoted are samples only.

For the sake of capsule exposition let us agree on the following terminology, which is mostly the usage in the book. If G is a connected simply-connected noncompact Lie group, if D is a discrete subgroup of G such that the homogeneous space G/D is compact, if $\phi: T \to G$ is a one-parameter subgroup of G where T is the line group, and if the action $\phi^*: T \times G/D \to G/D$ is defined by $\phi^*(t, gD) = \phi(t) \cdot gD$, then the flow $(G/D, \phi^*)$ is called a *coset flow*. If moreover G is nilpotent or solvable, then the compact manifold G/D is called a *nilmanifold* or a *solvmanifold*, and the flow $(G/D, \phi^*)$ is called a *nil-flow* or a *solvflow*.

Chapter I, An Outline of Results on Solvmanifolds, by Louis Auslander; pp. 1–6. This is a clear expository account of those known theorems about solvable and nilpotent connected Lie groups and their homogeneous spaces which are useful in the sequel. No proofs are given.

Chapter II, Ergodic Theory and Group Representations, by L. Green; pp. 7–14. This chapter brings together known definitions and theorems from ergodic theory and group representations which are needed later. The group representation theory here presented is useful in the study of ergodic properties of coset flows. The methods of Gelfand and Fomin (1952) and Mautner (1957) in this connection are both employed in the book. All irreducible unitary representations of two specific groups are described, the results for one group being new to the literature. Some proofs are included.

Chapter III, Flows on Some Three-Dimensional Homogeneous Spaces, by L. Auslander, L. Green, and F. Hahn; pp. 15–36. This section gives full answers to the questions: (1) to find all three-dimensional connected simply-connected noncompact Lie groups G which have a discrete subgroup D such that G/D is compact; (2) for each such group G to find all discrete subgroups D such that G/D is compact; (3) for the groups in (1) and (2) to describe the basic dynamical (that is, the topological and ergodic) properties of the coset flows on G/D and in particular to find which are transitive (some orbit is dense), which are minimal (every orbit is dense), which are ergodic, which are mixing. The groups in (1) fall into three classes: simple (just one), nilpotent, solvable but not nilpotent. Because of its particular historical association, we quote below a theorem about the simple case.

The geodesic and horocycle flows over compact surfaces of constant negative curvature have been studied with respect to their dynamical properties in considerable detail, particularly by Hedlund and by E. Hopf in the latter 1930's. It was more recently noted that such flows are special instances of flows on homogeneous spaces induced by one-parameter subgroups. The authors now place earlier work on these flows in a more general context by the following theorem:

Let G be the three-dimensional connected simply-connected noncompact simple Lie group (that is, let G be the universal covering group of the group of all two-by-two real matrices of determinant one), let D be a discrete subgroup of G such that G/D is compact, let X be a nonzero element of the Lie algebra of G which element is realized as a two-by-two real matrix of trace zero, and consider the coset flow on G/D induced by the one-parameter subgroup $\exp(Xt)$ of G. Then:

(a) (Generalized geodesic flow). If X has real distinct eigenvalues, then the coset flow on G/D is ergodic, strongly mixing, and has infinitely many periodic orbits.

(b) (Generalized horocycle flow). If X has real equal eigenvalues, then the coset flow on G/D is ergodic, strongly mixing, and minimal.

(c) (Periodic flow). If X has imaginary eigenvalues, then the coset flow on G/D is periodic.

Chapter III, Appendix, On the Fundamental Group of Certain Fibre Spaces, by W. Massey; pp. 37–41. In this excursus from the main theme of the book several theorems on bundles are proved which yield a method, useful in Chapter III, of determining the fundamental group of the bundle of unit tangent vectors to a compact orientable surface.

Chapter IV, Minimal Flows on Nilmanifolds, by L. Auslander, F. Hahn, and L. Markus; pp. 43–58. A flow is *distal* in case different points never get arbitrarily close as time varies. It is proved that all nilflows are distal and therefore, by a theorem of Robert Ellis, pointwise almost periodic. Furthermore, for every nilmanifold G/D there are many one-parameter subgroups ϕ of G such that the nilflow $(G/D, \phi^*)$ is minimal, that is, every orbit is dense. Now if a compact manifold is not a torus, then each minimal flow on the manifold is not equicontinuous. Thus a large class of nonequicontinuous distal minimal flows is produced. The phase spaces are compact analytic manifolds and the actions are analytic. This result shows in particular that many manifolds can carry minimal flows. These remarkable examples further strengthen the belief that hypotheses of geometric and analytic smoothness do not in general significantly reduce the topological types of flows. Incidentally, it is very much an open question as to what manifolds can carry minimal flows, even in the three-dimensional case.

Chapter V, Nilflows, Measure Theory, by L. Green; pp. 59–66. The measure-theoretic behavior of nilflows is studied here and a complete qualitative description of the spectra of nilflows is obtained. A nilflow is ergodic if and only if the associated torus flow is ergodic.

Minimal nilflows which are not toral have mixed spectrum. Group representations are used heavily in proofs.

Chapter VI, Flows on Certain Solvmanifolds not of Type E, by L. Auslander and F. Hahn; pp. 67–72. Type E means the exponential map is onto. A phenomenon occurring in Chapter III for dimension three is studied in greater generality.

Chapter VII, Flows in Type (E) Solvmanifolds, by L. Green; pp. 73–78. It is proved that a solvflow $(G/D, \phi^*)$, where G is of type (E), is ergodic if and only if an associated nilflow is ergodic. Group representations are used in the proof.

Chapter VII, Appendix, by L. Auslander; pp. 79–80. A "service" theorem is proved which is needed in Chapter VII.

Chapter VIII, An Application of Nilflows to Diophantine Approximations, by L. Auslander and F. Hahn; pp. 81–84. Let n be a positive integer and let I denote the set of the first n positive integers. Let $p_i(x) = \sum_{j\in I} a_{ij}x^j$ $(i\in I)$ be polynomials with integer coefficients, let $\sum_{j\in I} |a_{ij}| > 0$ $(i\in I)$, let λ_i $(i\in I)$ be real numbers such that $1, \lambda_1, \cdots, \lambda_n$ are rationally independent, let $\epsilon > 0$, and let θ_i $(i\in I)$ be arbitrary real numbers. Then there exists a relatively dense set P of integers such that $|\lambda_i p_i(c) - \theta_i| < \epsilon$ (mod 1) for all $i\in I$ and all $c\in P$. The theorem is proved by studying a particular nilflow.

Chapter IX, Discrete Groups with Dense Orbits, by Leon Greenberg; pp. 85–103. Let V be a finite-dimensional vector space over the real or complexes or quaternions, let G be a locally compact group of linear transformations of V such that G is strongly transitive (that is, sufficiently large in a certain precise sense), and let Γ be a discrete subgroup of G such that G/Γ is compact. Then it is proved by a very detailed study of eigenvalues that the orbit under Γ of every nonzero vector in V is dense in V. A theorem on horocycle flows (Hedlund, 1936) is a corollary. Also a theorem and several conjectures on values of forms (Mahler, 1959) are consequences.

There is a bibliography to the entire book of 57 items. A scanning of the bibliography alone will indicate the wide range of the work.

The mathematical power used in the book to analyze the global features of coset flows is of no small order. General theorems enter at a significant number of junctures to help overcome the difficulties. It seems clear that the import of this book will not only be to enrich the literature with many new welcome fascinating examples but also to encourage the production of general theorems which assist in the study of broad classes of examples.

The book is fully and carefully written with much attention to necessary factual background. Because of the inherent difficulty of the material not all parts are easy to read. Experts and students in

Lie groups or group representations will be much interested to observe here substantial applications of these theories and indeed some new contributions to these theories. The book is to be well recommended to many mathematicians on many counts but it is an absolute requirement as a source of inspiration to "live-wire topological dynamos" and "spark-plug measurable transformers."

WALTER H. GOTTSCHALK

Elementary methods in analytic number theory.[1] By A. E. Gelfond and U. V. Linnik. Moscow, 1964.

For many years, the theory of numbers has been expanding and growing by leaps and bounds. A great many interesting and significant results have been found, and important developments have been taking place. In due course, brief accounts of these are to be found in the mathematical reviews published in the U.S.A., Germany and the U.S.S.R., and later in encyclopedia articles. However, it has become increasingly difficult for any one to keep in close touch with all that is being done and published in various scattered journals and books.

There are three principal ways in which the new results become more accessible to the reader for closer study. For one, a body of knowledge dealing with closely related and connected results arises and these may be embodied in a systematic treatise. Thus there are Cassel's book on the Geometry of Numbers, the books on Transcendental Numbers by Siegel, Gelfond, and Schneider, Prachars' book on Prime Numbers, Walfisz' book on Exponential Sums.

Next there are books dealing with more loosely connected topics such as Delone and Faddeev's book on *Irrationalities of the third degree*, Vinogradov's book on the *Method of trigonometric sums in the theory of numbers*, and Lang's *Diophantine geometry.*

Finally, the book may contain a collection of miscellaneous topics which are mostly unrelated to each other, but which for various reasons make a special appeal to the author. Khintchine's *Three Pearls* is an instance of this.

The authors of the present book have followed the last pattern and in the twelve chapters of the book have assembled a collection of results representing many different and important aspects of number theory. Some of these results are well known but others have not appeared as yet in treatises. The reader will find a real treasure trove in this book.

Let us glance at the table of contents:

Chapter 1—Additive properties of numbers, the method of Schnirel-

[1] Rand, McNally and Co. are preparing an English translation.