ERGODIC PROPERTIES OF ISOMETRIES IN L^p SPACES, $1<p<\infty$

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Let $X=[0, 1]$, $\mathcal{B}$ the σ-algebra of Lebesgue measurable sets of X and μ the Lebesgue measure. For $1 \leqq p < \infty$ denote with $\mathcal{L}^p$ the vector space of all real-valued $\mathcal{B}$-measurable functions f on X for which $|f|^p$ is integrable and with $f \to \|f\|_p = (\int |f|^p d\mu)^{1/p}$ the corresponding seminorm on $\mathcal{L}^p$. Denote with $\mathcal{L}^\infty$ the vector space of all real-valued $\mathcal{B}$-measurable functions on X which are essentially bounded and with $f \to \|f\|_\infty$ the essential supremum seminorm on $\mathcal{L}^\infty$. For each $1 \leqq q \leqq \infty$ denote with L^q the associated Banach space and with $f \to \tilde{f}$ the canonical mapping of $\mathcal{L}^q$ onto L^q. If $T: L^q \to L^q$ is a continuous linear operator and $\tilde{f} \in L^q$, we shall denote by Tf a representative of the class $T\tilde{f}$. We shall say that *the individual ergodic theorem holds for T* if for every $f \in \mathcal{L}^q$

$$\lim_{m \to \infty} \frac{f(x) + Tf(x) + \cdots + T^{m-1}f(x)}{m}$$

exists almost everywhere. We shall say that *the dominated ergodic theorem holds for T* if there is a constant $C>0$ such that for every $f \in \mathcal{L}^q$

$$\sup_{1 \leqq m < \infty} \frac{|f + Tf + \cdots + T^{m-1}f|}{m} \in \mathcal{L}^q$$

and

$$\left\| \sup_{1 \leqq m < \infty} \frac{|f + Tf + \cdots + T^{m-1}f|}{m} \right\|_q \leqq C\|f\|_q.$$

Let us recall that an *automorphism* is a bijective mapping $\tau: X \to X$ satisfying the following two conditions: (i) for every $E \in \mathcal{B}$, $\tau^{-1}(E) \in \mathcal{B}$ and $\tau(E) \in \mathcal{B}$; (ii) if $A \in \mathcal{B}$ and $\mu(A)=0$, then $\mu(\tau^{-1}(A)) = \mu(\tau(A)) = 0$. Let $\mathcal{A}$ be the group of all automorphisms, e the unit element of $\mathcal{A}$ (i.e. the identity mapping of X). For $\tau_1 \in \mathcal{A}$, $\tau_2 \in \mathcal{A}$, write $\tau_1 \equiv \tau_2$ if $\mu(\{x \mid \tau_1(x) \neq \tau_2(x)\}) = 0$; this defines an equivalence relation R in $\mathcal{A}$. Denote with $\tau \to \tilde{\tau}$ the canonical mapping of the group $\mathcal{A}$ onto the quotient group $\mathcal{A}/R$.

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An automorphism τ is called *periodic* if there is $n \geqq 1$ such that $\tau^n \equiv e$.

Assume $1 \leqq q \leqq \infty$. For each $\tau \in \mathfrak{A}$ denote with T_τ the isometry of L^q (in this note the term isometry stands for a linear operator which preserves the norm) *induced* by τ; T_τ is defined by the equations

$$T_\tau f = f \circ \tau^{-1} \cdot \left(\frac{d\mu \circ \tau^{-1}}{d\mu} \right)^{1/q}, \qquad f \in \mathfrak{L}^q$$

(if $q = \infty$ the above equations reduce to $T_\tau f = f \circ \tau^{-1}, f \in \mathfrak{L}^\infty$). Remark that if $\tau' \in \mathfrak{A}$ and $\tau' \equiv \tau$, then $T_{\tau'} = T_\tau$. Denote with $\mathcal{G}_q$ the group of all isometries of L^q induced by automorphisms $\tau \in \mathfrak{A}$; the mapping $\tilde{\tau} \to T_\tau$ is an *isomorphism* of the group $\mathfrak{A}/R$ onto the group $\mathcal{G}_q$ (for each $1 \leqq q \leqq \infty$).

It is known that *the individual ergodic theorem does not hold in general for $T \in \mathcal{G}_1$ in L^1 and $T \in \mathcal{G}_\infty$ in L^∞, respectively* (see [2; 5; 6; 9]). The purpose of this note is to show that if $1 < p < \infty$, then a dominated ergodic theorem (with the best possible constant) holds for any $T \in \mathcal{G}_p$ in L^p and that in a certain sense this result can be obtained from the classical dominated ergodic theorem for measure-preserving automorphisms; as a consequence we deduce that the individual ergodic theorem holds for any $T \in \mathcal{G}_p$ in L^p.[2]

Throughout the rest of the paper we shall suppose p fixed and such that $1 < p < \infty$; from now on we shall omit the subscript p from the group $\mathcal{G}_p$ and will write $\mathcal{G}$ instead of $\mathcal{G}_p$. By a theorem of Banach (see [1, p. 178] and [7])[3] every positive invertible isometry of L^p is induced by some automorphism $\tau \in \mathfrak{A}$; thus *the group of all positive invertible isometries of L^p coincides with* $\mathcal{G}$. We shall consider on $\mathcal{G}$ the topology $\mathfrak{I}$ induced by the strong operator topology; endowed with $\mathfrak{I}$, $\mathcal{G}$ is a topological group.

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[2] The author is grateful to Dr. E. M. Stein for bringing this problem to her attention. In fact Stein had obtained a dominated ergodic theorem for the case $p = 2$ by a method similar in spirit to that of Theorem 2 in his paper [8]; he conjectured that a dominated ergodic theorem probably holds for any $1 < p < \infty$ and raised the question whether or not this could be derived from the classical ergodic theorem for measure-preserving automorphisms.

[3] The various versions of Banach's theorem found in the literature are given for $p \neq 2$ and arbitrary (not necessarily positive) isometries of L^p. If $p = 2$ however, then it is easily seen that every *positive* invertible isometry of L^2 is also induced by some automorphism. It is sufficient to remark that T has the following crucial property: (α) If $f \in \mathfrak{L}^2$, $g \in \mathfrak{L}^2$ and $f(x) \cdot g(x) = 0$ almost everywhere, then $Tf(x) \cdot Tg(x) = 0$ almost everywhere. From property (α) we deduce as in the proof of Theorem 3.1 in [7] that T is induced by an ≪automorphism of the measure algebra≫. Since our measure space is a Lebesgue space, we infer that T is actually induced by some (point) automorphism $\tau \in \mathfrak{A}$.

Denote with $\mathcal{P}$ the set of all $T=T_\tau\in\mathcal{G}$ for which τ is periodic. We then have:

(P) *The set $\mathcal{P}$ is dense in $\mathcal{G}$.*

Proposition (P) can be proved for instance in the same way as Proposition 3 in [6], using Linderholm's approximation theorem (given $\tau\in\mathcal{A}$ and $\epsilon>0$ there is $\zeta\in\mathcal{A}$ periodic such that $\mu(\{x\mid\tau(x)\neq\zeta(x)\})\leqq\epsilon$).

THEOREM. *Let T be a positive invertible isometry of $L^p=L^p(X,\mathcal{B},\mu)$. Then for each $f\in\mathcal{L}^p$ we have*

$$(*)\qquad \|f^*\|_p\leqq\left(\frac{p}{p-1}\right)\|f\|_p;$$

here

$$f^*(x)=\sup_{1\leqq m<\infty}\frac{|f(x)+Tf(x)+\cdots+T^{m-1}f(x)|}{m},$$

for $x\in X$.

PROOF. We shall divide the proof into three parts. We shall first prove the following result:

(A) *Let $\tau\in\mathcal{A}$ and $\nu\geqq 0$ a totally σ-finite measure on $\mathcal{B}$ equivalent to μ. Let S be the isometry of $L^p(X,\mathcal{B},\mu)$ induced by τ ($Sf=f\circ\tau^{-1}\cdot(d\mu\circ\tau^{-1}/d\mu)^{1/p}$ for $f\in\mathcal{L}^p(X,\mathcal{B},\mu)$) and U the isometry of $L^p(X,\mathcal{B},\nu)$ induced by τ ($Ug=g\circ\tau^{-1}\cdot(d\nu\circ\tau^{-1}/d\nu)^{1/p}$ for $g\in\mathcal{L}^p(X,\mathcal{B},\nu)$). The dominated ergodic theorem holds for S in $L^p(X,\mathcal{B},\mu)$ if and only if the dominated ergodic theorem holds for U in $L^p(X,\mathcal{B},\nu)$, the constant being the same.*

We shall only prove one implication, the proof of the converse implication being entirely similar. Assume that the dominated ergodic theorem holds for S in $L^p(X,\mathcal{B},\mu)$ with the constant $C>0$; we shall show that the dominated ergodic theorem holds for U in $L^p(X,\mathcal{B},\nu)$ with the constant C.

Let $h=d\nu/d\mu$ and remark that for each $n\geqq 0$ we have

$$(1)\qquad \frac{d\nu\circ\tau^{-n}}{d\nu}\equiv\frac{d\nu\circ\tau^{-n}}{d\mu\circ\tau^{-n}}\cdot\frac{d\mu\circ\tau^{-n}}{d\mu}\cdot\frac{d\mu}{d\nu}\equiv h\circ\tau^{-n}\cdot\frac{d\mu\circ\tau^{-n}}{d\mu}\cdot\frac{1}{h}\,.$$

Let now $g\in\mathcal{L}^p(X,\mathcal{B},\nu)$; then $gh^{1/p}\in\mathcal{L}^p(X,\mathcal{B},\mu)$ and we deduce from (1) that

$$(2)\qquad U^ng=g\circ\tau^{-n}\cdot\left(\frac{d\nu\circ\tau^{-n}}{d\nu}\right)^{1/p}\equiv\frac{1}{h^{1/p}}\cdot S^n(gh^{1/p})\qquad\text{for each } n\geqq 0.$$

If we use the notation $U_m = (I + U + \cdots + U^{m-1})/m$, $S_m = (I+S+ \cdots +S^{m-1})/m$ for each $1 \leqq m < \infty$ we deduce that

$$U_m g \equiv \frac{1}{h^{1/p}} \cdot S_m(gh^{1/p}) \qquad \text{for each } 1 \leqq m < \infty;$$

hence

$$\sup_{1 \leqq m < \infty} | U_m g | \equiv \frac{1}{h^{1/p}} \cdot \sup_{1 \leqq m < \infty} | S_m(gh^{1/p}) |. \tag{3}$$

From (3) it follows that $\sup_{1 \leqq m < \infty} | U_m g|$ belongs to $\mathfrak{L}^p(X, \mathfrak{B}, \nu)$ and that

$$\begin{aligned} \int \Big(\sup_{1 \leqq m < \infty} | U_m g | \Big)^p d\nu &= \int \frac{1}{h} \Big(\sup_{1 \leqq m < \infty} | S_m(gh^{1/p}) | \Big)^p d\nu \\ &= \int \Big(\sup_{1 \leqq m < \infty} | S_m(gh^{1/p}) | \Big)^p d\mu \leqq C^p \Big(\int | gh^{1/p} |^p d\mu \Big) \\ &= C^p \Big(\int | g |^p h d\mu \Big) = C^p \Big(\int | g |^p d\nu \Big). \end{aligned}$$

This shows that the dominated ergodic theorem holds for U in $L^p(X, \mathfrak{B}, \nu)$ with the constant C.

We shall prove next that:

(B) *For every $T \in \mathcal{P}$ the dominated ergodic theorem holds in $L^p(X, \mathfrak{B}, \mu)$ with the constant $p/p-1$.*

Let $T = T_\tau \in \mathcal{P}$. Since τ is periodic, there is $n \geqq 1$ such that $\tau^n \equiv e$. Define the measure ν on $\mathfrak{B}$ by the equation $\nu = \sum_{0 \leqq j \leqq n-1} \mu \circ \tau^j$; ν is a finite measure on $\mathfrak{B}$ equivalent to μ and ν is invariant under τ. The result (B) is then an immediate consequence of the classical dominated ergodic theorem for measure-preserving automorphisms and of proposition (A).

(C) *We shall now conclude the proof of the theorem.* Let $T \in \mathcal{G}$ be arbitrary and let $f \in \mathfrak{L}^p$. Since T is a positive operator, we may assume without loss of generality that $f \geqq 0$. To prove the inequality (*) for f it will be enough to show that for any $k \geqq 1$, any disjoint sets $A_1 \in \mathfrak{B}, \cdots, A_k \in \mathfrak{B}$ and any integers $n_1 \geqq 1, \cdots, n_k \geqq 1$ we have:

$$\Big\| \sum_{1 \leqq j \leqq k} \phi_{A_j} \frac{f + Tf + \cdots + T^{n_j - 1} f}{n_j} \Big\|_p \leqq \Big(\frac{p}{p-1} \Big) \|f\|_p. \tag{4}$$

But by (B) the inequality (4) holds if we replace T by an arbitrary isometry belonging to $\mathcal{P}$. Since $\mathcal{P}$ is dense in $\mathcal{G}$ (see Proposition (P) above), an elementary argument shows that (4) holds for T. Thus the proof of the theorem is complete.

COROLLARY. *If T is a positive invertible isometry of $L^p = L^p(X, \mathcal{B}, \mu)$ then the individual ergodic theorem holds for T.*

PROOF. Consider the vector spaces

$$\mathfrak{N} = \{\tilde{f} \mid T\tilde{f} = \tilde{f}\}$$

and

$$\mathfrak{M} = (I - T)L^p = \{\tilde{g} \mid \tilde{g} = (I - T)\tilde{h} \text{ for some } \tilde{h} \in L^p\}.$$

By the mean ergodic theorem the set (direct sum) $\mathfrak{N} + \mathfrak{M}$ is dense in L^p. Now for each $\tilde{f} \in \mathfrak{N} + \mathfrak{M}$,

$$\lim_{m\to\infty} \frac{f(x) + Tf(x) + \cdots + T^{m-1}f(x)}{m}$$

exists almost everywhere; to see this it is enough to use the fact[4] that for each $h \in \mathcal{L}^p$, $\lim_{m\to\infty} T^m h(x)/m = 0$ almost everywhere.

Since by the above theorem,

$$\sup_{1 \leq m < \infty} \frac{|f(x) + Tf(x) + \cdots + T^{m-1}f(x)|}{m} < \infty$$

almost everywhere, for each $f \in \mathcal{L}^p$, the Banach convergence theorem can be applied (see [**3**, pp. 332–334]) and we conclude that the individual ergodic theorem holds for T.

REMARKS. (1) For $p \neq 2$ the above theorem and corollary remain valid for an arbitrary (not necessarily positive) invertible isometry of L^p, since by the theorem of Banach (see [**1**, p. 178] and [**7**]) every such isometry T is "dominated" by a positive invertible isometry S (in the sense that for each $f \in \mathcal{L}^p$, $|Tf(x)| \leq S|f|(x)$ almost everywhere). (2) Proposition (A) given in the course of the proof of the above theorem remains valid (with obvious modifications) if we replace the Lebesgue space $(X, \mathcal{B}, \mu)$ by an arbitrary totally σ-finite measure space and τ by an ≪automorphism of the corresponding measure algebra≫. (3) The theorem, the corollary and Remark (1) above remain valid if we replace the Lebesgue space $(X, \mathcal{B}, \mu)$ by an arbitrary totally σ-finite measure space (consider a finite equivalent measure, apply the generalized version of Banach's theorem on

[4] This follows from the obvious inequality $\sum_{m=1}^{\infty} \|(T^m h)/m\|_p^p \leq \|h\|_p^p (\sum_{m=1}^{\infty} 1/m^p) < \infty$ which is in fact true for an arbitrary contraction T of L^p and any $h \in \mathcal{L}^p$, as was remarked by Mr. Ackoglu from Brown University.

isometries given by Lamperti (see [7, pp. 461–463]) and make use of the isomorphism of a separable nonatomic normalized measure algebra with the measure algebra of the unit interval (see [4, p. 173])).

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