ON THE RATE OF GROWTH OF ENTIRE FUNCTIONS OF FAST GROWTH

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1. **Introduction.** The purpose of this note is to generalize the following well-known formula to give the order ρ and type σ of an entire function $f(z)=\sum_{n=0}^{\infty} a_n z^n$ with $M(r)=\max_{|z|=r} |f(z)|$, that is [1; 2],

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} = \limsup_{n\to\infty} \frac{n\log n}{-\log |a_n|}, \tag{1}$$

$$\sigma = \limsup_{r\to\infty} \frac{\log M(r)}{r^{\rho}} = \frac{1}{e\rho}\cdot \limsup_{n\to\infty} n |a_n|^{\rho/n}. \tag{2}$$

It will be observed that the coefficient $1/(e\rho)$ in (2) comes exclusively into the case of entire functions of finite order as we will see in the Theorem I.

2. **Definitions. Notations and preparatory lemmas.**

NOTATION 1. $\exp^{[0]} = \log^{[0]}x = x$, $\exp^{[m]}x = \log^{[-m]}x = \exp(\exp^{[m-1]}x) = \log(\log^{[-m-1]}x)$ $(m=0, \pm 1, \pm 2, \cdots)$.

NOTATION 2.

$$E_{[r]}(x) = \prod_{i=0}^{r} \exp^{[i]} x, \qquad \Lambda_{[r]}(x) = \prod_{i=0}^{r} \log^{[i]} x,$$

$$E_{[-r]}(x) = x/\Lambda_{[r-1]}(x), \qquad \Lambda_{[-r]}(x) = x/E_{[r-1]}(x),$$

$$x = E_{[r]}^{[-1]}(y) \Leftrightarrow y = E_{[r]}(x) \qquad (r = 0, \pm 1, \pm 2, \cdots).$$

LEMMAS. *The functions* $\exp^{[m]}x$, $\log^{[m]}x$, $E_{[r]}(x)$, $\Lambda_{[r]}(x)$, $E_{[r]}^{[-1]}(x)$ $(m=0, \pm 1, \pm 2, \cdots; r=0, 1, 2, \cdots)$ *all increase monotonically and we have*

$$\frac{d}{dx}(\exp^{[m]} x) = \frac{E_{[m]}(x)}{x} = \frac{1}{\Lambda_{[-m-1]}(x)}, \tag{3}$$

$$\frac{d}{dx}(\log^{[m]} x) = \frac{1}{\Lambda_{[m-1]}(x)} = \frac{E_{[-m]}(x)}{x} \qquad (m = 0, \pm 1, \pm 2, \cdots) \tag{4}$$

$$E_{[r]}^{[-1]}(y) = \begin{cases} y & (r = 0) \\ \log^{[r-1]}(\log y - \log^{[2]} y + O(\log^{[3]} y)) & (r = 1, 2, 3, \cdots) \end{cases} \tag{5}$$

$$(6)\qquad \lim_{y\to\infty} \exp(E_{[1-q]}(y)) = \begin{cases} e & (q = 2), \\ 1 & (q = 3, 4, 5, \cdots). \end{cases}$$

$$(7)\qquad \lim_{y\to\infty} (\exp^{[q-1]}(E^{[-1]}_{[q-2]}(y)))^{1/y} = \begin{cases} e & (q = 2), \\ 1 & (q = 3, 4, 5, \cdots). \end{cases}$$

DEFINITION 1. Given an entire function $f(z)$ with $M(r)$ defined in §1, then we define the *Lambda of index q* by

$$(8)\qquad \lambda_{(q)} = \limsup_{r\to\infty} \frac{\log^{[q]} M(r)}{\log r} = \lambda$$

and when $0<\lambda<\infty$, then define *Kappa of index q* by

$$(9)\qquad \kappa_{(q)} = \limsup_{r\to\infty} \frac{\log^{[q-1]} M(r)}{r^{\lambda}} \cdot$$

DEFINITION 2. An entire function $f(z)$ with $\lambda_{(q-1)} = \infty$ and $\lambda_{(q)} < \infty$ is called an *entire function of index q*. The entire function of index 0 is the constant function. The entire function of index 1 is a rational entire function in which $\lambda_{(1)}$ is its degree and $\kappa_{(1)}$ is the magnitude of its leading coefficient. The entire function of index 2 is called the transcendental entire function of finite order in which $\lambda_{(2)}$ is called the order, and $\kappa_{(2)}$ is called the type. $\lambda_{(3)}$ is called the rank and $\kappa_{(3)}$ is called the title of the entire function. We call $\lambda_{(q)}$ and $\kappa_{(q)}$ the rate of growth of the entire function of index q.

3. Formulas for $\lambda_{(q)}$ and $\kappa_{(q)}$.

THEOREM I. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a transcendental entire function of index q, then $\lambda_{(q)} = \lambda$ and $\kappa_{(q)} = \kappa$ of $f(z)$ is given by $\lambda = \mu$ and $\kappa = \tau$ where*

$$(10)\qquad \mu = \limsup_{n\to\infty} \frac{n \log^{[q-1]} n}{-\log |a_n|} \qquad (q = 2, 3, 4, \cdots)$$

and

$$(11)\qquad \tau = \begin{cases} (1/e\lambda) \cdot \limsup_{n\to\infty} n |a_n|^{\lambda/n} & (q = 2) \\ \limsup_{n\to\infty} \log^{[q-2]} n \cdot |a_n|^{\lambda/n} & (q = 3, 4, 5, \cdots). \end{cases}$$

PROOF. From $-n \log^{[q-1]} n / \log |a_n| \leqq \mu + \epsilon$, we have, with $S = \exp^{[q-2]}((2r)^{\mu+\epsilon})$, that

$$\begin{aligned} M(r) &\leqq \sum_{n\leqq s} |a_n| r^n + \sum_{n>s} |a_n| r^n \\ (12)\qquad &\leqq \exp^{[q-1]}((2r)^{\mu+2\epsilon}) \cdot \sum_{n=0}^{\infty} (\log^{[q-2]} n)^{-n/(\mu+\epsilon)} + \sum_{n=0}^{\infty} \frac{1}{2^n} \\ &= O(\exp^{[q-1]} r^{\mu+3\epsilon}). \end{aligned}$$

Letting $\epsilon \to 0$ we have $\lambda \leqq \mu$.

Let $\sigma = e\lambda\tau$ for $q=2$ and $\sigma=\tau$ for $q \geqq 3$, from

$$|a_n|^{\lambda/n} \cdot \log^{[q-2]} n \leqq \sigma + \epsilon \tag{13}$$

we see, by logarithmic differentiation with (4), that the maximum of $|a_n| r^n$ is estimated by

$$|a_n| r^n \leqq \exp\left(\left(\exp^{[q-2]}\left(\frac{(\sigma+\epsilon)r^\lambda}{\exp(E_{[1-q]}(n))}\right)\right)\cdot\frac{E_{[1-q]}(n)}{\lambda}\right) \equiv \phi(r). \tag{14}$$

Hence, we have, with $s' = \exp^{[q-2]}((\sigma+2\epsilon)r^\lambda)$, using (6), that

$$\begin{aligned} M(r) &\leqq \sum_{n \leqq s'} |a_n| r^n + \sum_{n > s'} |a_n| r^n \\ &\leqq s'\phi(r) + \sum_{n=0}^{\infty}\left(\frac{\sigma+\epsilon}{\sigma+2\epsilon}\right)^{n/\lambda} \\ &= O(\exp^{[q-1]}((\tau+3\epsilon)r^\lambda)). \end{aligned} \tag{15}$$

Letting $\epsilon \to 0$, we have $\kappa \leqq \tau$. Suppose now, that $M(r) < C \exp^{[q-1]}((\kappa+\epsilon)r^\lambda)$ then $|a_n| < M(r)/r^n$ is estimated by minimizing its right hand side which occurs, by (3), at $r = (E^{[-1]}_{[q-2]}(n/\lambda)/(\kappa+\epsilon))^{1/\lambda}$. Hence, we have

$$|a_n| < \frac{C(\kappa+\epsilon)^{n/\lambda} \cdot \exp^{[q-1]}(E^{[-1]}_{[q-2]}(n/\lambda))}{(E^{[-1]}_{[q-2]}(n/\lambda))^{n/\lambda}} \tag{16}$$

from which we have by (5) and (7), that

$$\lambda \geqq \limsup_{n\to\infty} \frac{n \log^{[q-1]} n}{-\log |a_n|} = \mu \tag{17}$$

and

$$\kappa + \epsilon \geqq \frac{\tau}{\sigma} \cdot \limsup_{n\to\infty} \log^{[q-2]} n \cdot |a_n|^{\lambda/n} = \tau. \tag{18}$$

The theorem is thereby proved.

4. Further remarks.

1. *Utterly integer valued transcendental entire function.* We have many results on the integer valued entire functions of index $q=2$, (finite order) i.e., [3] but here we introduce a theorem on index $q \geqq 3$, whose proof together with its generalization and applications on number theory will appear in a future paper.

THEOREM II. *A transcendental entire function which together with all its derivatives assumes integers at all integer points (utterly integer* *valued) must have index $q \geqq 3$ and, if the index is 3, then its rank must be $\lambda_{(3)} \geqq 1$. This estimation is the best possible one, since there exist such a transcendental entire function of index 3 and $\lambda_{(3)} = 1$.*

2. *Entire function of infinite index.* For any positive increasing function $\psi(n)$ with $\psi(n) \to \infty$ as $n \to \infty$, but for no function with $\liminf_{n\to\infty} \psi(n) = m < \infty$, the series $f(z) = \sum_{n=0}^{\infty} z^n/(\psi(n))^n$ represents an entire function, hence if $\psi(n)$ grows slower than any $\log^{[N]} n$ with fixed N, then $f(z)$ represents an entire function of infinite index. To define the rate of growth, the natural comparison function will be $\phi(r) = f((\alpha+\epsilon)r^{\beta})$ with $f(x) = \exp^{[x]} 1$, $[x]$: Gauss step function.

3. *Entire functions of nonintegral index.* Consider $f(x) = \exp^{(q/p)} x$ as a well defined solution of a simultaneous functional equation $\exp^{[tp]}(f(x)) = \exp^{[tq]} x$ $(t = 0, \pm 1, \pm 2, \cdots)$ and for real r, define $\exp^{(r)}(x)$ by uniform limit process. Generalize an index as the least number η such that for any given $\epsilon > 0$, there exist $r_0(\epsilon)$ by which it satisfies $M(r) < \exp^{(\eta+\epsilon)}(r)$ for $r \geqq r_0(\epsilon)$, when $\eta < \infty$, define $\lambda_{(\eta)}$ and $\kappa_{(\eta)}$ by the similar manner.

The author conjectures to have the similar formula as in Theorem I, but this formulation is incomplete at this moment.

5. **A research problem.** To generalize the discussion into the meromorphic functions, we propose the following problem which is originally given by E. G. Straus.

Problem. Let $f(z)$ be a meromorphic function and $T(r)$ be its characteristic function, let

$$\lambda_{(q)} = \limsup_{r\to\infty} \frac{\log^{[q-1]} T(r)}{\log r} = \lambda \tag{19}$$

and, when $0 < \lambda < \infty$,

$$\kappa_{(q)} = \limsup_{r\to\infty} \frac{\log^{[q-2]} T(r)}{r^{\lambda}} = \kappa. \tag{20}$$

Find the formula to give λ and κ from the Taylor series coefficients of $f(z)$.

References

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