ISOMETRIC FLOWS ON HILBERT SPACE

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1. **Introduction.** It is known that if V is an isometry on a (complex) Hilbert space $\mathfrak{X}$ onto a subspace R of $\mathfrak{X}$, then

$$(1.1)\qquad \mathfrak{X} = \sum_{k=0}^{\infty} V^k(R^\perp) + \bigcap_{k=0}^{\infty} V^k(\mathfrak{X}),$$

where the two subspaces on the right-hand side are orthogonal, and $R^\perp$ is "wandering for V," i.e. $V^j(R^\perp) \perp V^k(R^\perp)$, $j \neq k$.[2] The identity **(1.1)** closely resembles the Wold decomposition of the "present and past subspace" of a weakly stationary stochastic process into its "innovation subspaces" and the "remote past" cf. [**10**, 6.10]. Interpreting k as the time, we shall therefore speak of (1.1) as the *Wold decomposition* of $\mathfrak{X}$ due to V or (equivalently) due to the discrete semi-group (V^k, $k \geqq 0$), and refer to $V^k(R^\perp)$, $k \geqq 0$, as the *innovation subspaces* of $\mathfrak{X}$, and to $\bigcap_{k=0}^{\infty} V^k(\mathfrak{X})$ as the *remote subspace* of $\mathfrak{X}$ engendered by the semi-group.

In this note our purpose is to obtain the analogous decomposition of $\mathfrak{X}$ due to a strongly continuous semi-group (S_t, $t \geqq 0$) of isometries on $\mathfrak{X}$ into $\mathfrak{X}$ (6.5 below). We shall derive this by applying (1.1) to the Cayley transform V of H, where iH is the infinitesimal generator of the semi-group, and then replacing the direct sum $\sum_{k=0}^{\infty} V^k(R^\perp)$ of innovation subspaces, occurring in (1.1), by a direct integral of "differential innovation subspaces."

2. **The associated discrete semi-group.** Let (S_t, $t \geqq 0$) be a strongly continuous semi-group of isometries on $\mathfrak{X}$ into $\mathfrak{X}$, and let iH be its infinitesimal generator. Then

$$(2.1)\qquad S_t' = S_t iH = iHS_t, \qquad \text{on } \mathfrak{D}, \qquad t \geqq 0,$$

where $\mathfrak{D}$, the domain of H, is a linear manifold everywhere dense in $\mathfrak{X}$. From the work of J. L. B. Cooper [**1**], (cf. also [**5**])[3] we know that

(a) *H is maximal symmetric with deficiency index* $(0, \alpha)$,

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[2] This result, implicit in the work of von Neumann and Murray on rings of operators, is proved and put to significant use in a recent paper by Halmos [4] (cf. also [6]).

[3] Our approach differs from Cooper's in that we make systematic use of the operator $T_{a,b}$ defined in (4.1), and of the deficiency subspace $R^\perp$ of H.

(b) *$H+iI$ is one-one on $\mathfrak{D}$ onto $\mathfrak{X}$,*

(2.2) (c) *$(H+iI)^{-1}=\frac{1}{i}\int_0^\infty e^{-t}S_t dt$ is one-one and bounded on $\mathfrak{X}$ onto $\mathfrak{D}$ and $|(H+iI)^{-1}|\leqq 1$,*[4]

(d) *$H-iI$ is one-one on $\mathfrak{D}$ onto a (closed) subspace R.*[5]

Now let V be the Cayley transform of H:

$$V = c(H) = (H - iI)(H + iI)^{-1}, \qquad \text{on } \mathfrak{X}.$$

It follows from the work of von Neumann, cf. [9, Chapter IX], that

$$(2.3)\quad \begin{aligned} &\text{(a) } V \textit{ is an isometry on } \mathfrak{X} \textit{ onto } R,\\ &\text{(b) } I - V = 2i(H + iI)^{-1} = 2\int_0^\infty e^{-t}S_t dt \qquad \textit{on } \mathfrak{X},\\ &\text{(c) } H = i(I + V)(I - V)^{-1} \textit{ on } \mathfrak{D},\\ &\text{(d) } S_tV^k = V^kS_t \qquad \textit{on } \mathfrak{X}, \qquad t \geqq 0,\ k \geqq 0. \end{aligned}$$

We shall call (V^k, $k\geqq 0$) the *discrete semi-group of isometries associated with the given semi-group* (S_t, $t\geqq 0$). In the rest of §2 we shall formulate the basic relationship between the S_t and the V^k.

The S_t are expressible in terms of H by the exponential formula, cf. [5],

$$(2.4)\quad \begin{aligned} S_t &= \lim_{n\to\infty} \exp(tiHJ_n), \qquad \textit{strongly on } \mathfrak{X},\\ \textit{where}\quad J_n &= \left(I - \frac{1}{n}iH\right)^{-1}. \end{aligned}$$

Since J_n is a bounded operator, so therefore is $iHJ_n = n(J_n - I)$. Hence $\exp(tiHJ_n)$ has a power series expansion, from which we get the following expression for S_t in terms of V^k:

$$(2.5)\quad \begin{aligned} S_t &= e^{-t}I + \lim_{n\to\infty}\sum_{k=1}^{\infty}\left\{\frac{1}{k!}\left(\frac{-nt}{n+1}\right)^k \sum_{j=1}^{k}\binom{k}{j}K_n^j\right\}, \qquad t \geqq 0,\\ K_n &= \frac{2n}{n+1}\left\{I - \frac{n-1}{n+1}V\right\}^{-1}V, \qquad \textit{and} \quad \textit{so} \quad K_n(\mathfrak{X}) \subseteq R, \quad n \geqq 1. \end{aligned}$$

Reciprocally, we find from (2.3)(b) the following expression for V^n in terms of the S_t:

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[4] The absolute value sign refers to the usual Banach norm of the operator.

[5] We will have $R=\mathfrak{X}$ if and only if H is self-adjoint, which in turn will be the case if and only if the isometries S_t are actually unitary. For us this is the uninteresting case in which the Wold decomposition reduces to the triviality $\mathfrak{X}=\mathfrak{X}$.

$$
(2.6)\qquad \begin{aligned} V^n &= I + 2\int_0^\infty L_n'(2t)e^{-t}S_t dt, \\ L_n(t) &= \sum_{k=0}^{n} \frac{(-1)^k}{k!}\binom{n}{k} t^k, \qquad (n\text{th Laguerre polynomial, [8]}). \end{aligned}
$$

From (2.5), (2.6) we get the following useful identity between the subspaces generated by the sets $S_t(X)$, $t \geqq 0$, and $V^k(X)$, $k \geqq 0$:

$$
(2.7)\qquad \mathfrak{S}\{S_t(X)\}_{t\geqq 0} = \mathfrak{S}\{V^k(X)\}_{k\geqq 0}, \qquad X \subseteq \mathfrak{X}.
$$

From (2.5) we also see that

$$
(2.8)\qquad S_t(x) = e^{-t}x + y_t, \qquad y_t \in R,\ t \geqq 0,\ x \in \mathfrak{X},
$$

and hence

$$
(2.9)\qquad \begin{aligned} (S_t(x), y) &= e^{-t}(x, y), \qquad x \in \mathfrak{X},\ y \in R^\perp,\ t \geqq 0, \\ (S_s(x), S_t(y)) &= e^{-|s-t|}(x, y), \qquad x, y \in R^\perp,\ s, t \geqq 0, \end{aligned}
$$

where (,) denotes the inner product in $\mathfrak{X}$.

3. **The remote subspace.** Let us write

$$
(3.1)\qquad \begin{aligned} \mathfrak{X}_t &= S_t(\mathfrak{X}), \qquad \mathfrak{X}_k' = V^k(\mathfrak{X}), \qquad t, k \geqq 0, \\ \mathfrak{X}_\infty &= \bigcap_{t\geqq 0} \mathfrak{X}_t, \qquad \mathfrak{X}_\infty' = \bigcap_{k\geqq 0} \mathfrak{X}_k'. \end{aligned}
$$

We assert the following crucial theorem:

3.2. THEOREM. $\mathfrak{X}_\infty = \mathfrak{X}_\infty'$. *The restrictions of the isometries S_t, V^n, for $t, n \geqq 0$, to the subspace $\mathfrak{X}_\infty$ are unitary.*

To prove this we first show, quite easily, that the restrictions of S_t and V^k to the remote subspaces $\mathfrak{X}_\infty$, $\mathfrak{X}_\infty'$, respectively, are unitary. We then establish the deeper result $\mathfrak{X}_\infty = \mathfrak{X}_\infty'$. The inclusion $\mathfrak{X}_\infty \subseteq \mathfrak{X}_\infty'$ follows without much difficulty from (2.8) and (2.3). The reverse inclusion requires the following lemma, which rests on the fact that $\mathfrak{D}$ is the range of $I - V$, and on the limiting behavior of $L_n(t)$, as $n \to \infty$, cf. [8, pp. 333–334]:

LEMMA. *Let $\mathfrak{D}_\infty' = \bigcap_{k\geqq 0} V^k(\mathfrak{D})$, where $\mathfrak{D}$ is the domain of the infinitesimal generator iH. Then*

(a) *$\mathfrak{D}_\infty'$ is a linear manifold everywhere dense in $\mathfrak{X}_\infty'$,*

(b) *$\mathfrak{D}_\infty' \subseteq \mathfrak{X}_\infty$ (and so $\mathfrak{X}_\infty' = \text{clos.}\mathfrak{D}_\infty' \subseteq \mathfrak{X}_\infty$).*

Let us take the Wold decomposition (1.1) of $\mathfrak{X}$ due to $V = c(H)$, the Cayley transform of H. As just shown $\bigcap_{k=0}^\infty V^k(\mathfrak{X}) = \mathfrak{X}_\infty$. Also, on taking $X = R^\perp$ in (2.7) we find that $\sum_{k=0}^\infty V^k(R^\perp) = \mathfrak{S}\{V^k(R^\perp)\}_{k\geqq 0} = \mathfrak{S}\{S_t(R^\perp)\}_{t\geqq 0}$. Thus (1.1) reduces to

$$\mathfrak{X} = \mathfrak{S}\{S_t(R^\perp)\}_{t\geqq 0} + \mathfrak{X}_\infty, \qquad \mathfrak{S}\{S_t(R^\perp)\}_{t\geqq 0} \perp \mathfrak{X}_\infty.$$

On applying S_a we get

$$(3.3) \quad \mathfrak{X}_a = \mathfrak{S}\{S_t(R^\perp)\}_{t\geqq a} + \mathfrak{X}_\infty, \ (a \geqq 0), \qquad \mathfrak{S}\{S_t(R^\perp)\}_{t\geqq 0} \perp \mathfrak{X}_\infty.$$

We shall refer to (3.3) as the *pre-Wold decomposition of* $\mathfrak{X}_a$ *due to the semi-group* $(S_t, t \geqq 0)$. Our task is to express the first subspace on the right-hand side as a direct integral of differential subspaces.

4. **Differential innovation subspaces.** We first introduce an *operator-valued interval-measure.* The measure T_{ab} of the interval $[a, b]$, $0 \leqq a \leqq b$, is defined by

$$(4.1) \quad T_{ab} = T_b - T_a, \ \textit{where} \ T_t = \frac{1}{\sqrt{2}}\left\{S_t - I - \int_0^t S_s ds\right\}, \quad t \geqq 0.$$

We see at once that T_{ab}, T_t are bounded linear operators on $\mathfrak{X}$ into $\mathfrak{X}$, that $T_t = T_{0t}$, $t \geqq 0$, and that

$$(4.2) \quad \begin{aligned} &\text{(a)} \ T_{ab} + T_{bc} = T_{ac}, && 0 \leqq a \leqq b \leqq c, \\ &\text{(b)} \ T_{ab} = \frac{1}{\sqrt{2}}\left\{S_b - S_a - \int_a^b S_s ds\right\}, && 0 \leqq a \leqq b \\ &\text{(c)} \ S_t T_{ab} = T_{a+t,b+t}, && 0 \leqq a \leqq b, \ 0 \leqq t. \end{aligned}$$

By inverting the relations (4.1) we get the following expression for S_t in terms of the $T_{\sigma\tau}$:

$$(4.3) \quad S_t = -\sqrt{2}\int_t^\infty e^{t-s} T_{ts} ds = \sqrt{2}\left\{T_t - \int_t^\infty e^{t-s} T_s ds\right\}.$$

We consider next the *subspace-valued interval measure*:

$$(4.4) \qquad \mathfrak{N}_{ab} = T_{ab}(R^\perp), \qquad 0 \leqq a \leqq b.$$

This has the following convenient properties, which are easy to check:

(4.5)

(a) $S_t(\mathfrak{N}_{ab}) = \mathfrak{N}_{a+t,b+t}$, $0 \leqq a \leqq b$, $0 \leqq t$;

(b) $\mathfrak{N}_{ab} \perp \mathfrak{N}_{cd}$, $0 \leqq a < b \leqq c < d$;

(c) $\dfrac{1}{\sqrt{(b-a)}} T_{ab}$ *is an isometry on* $R^\perp$ *onto* $\mathfrak{N}_{ab}$, $a < b$;

i.e. $(T_{ab}x, T_{ab}y) = (b-a)(x, y)$, $x, y \in R^\perp$;

(d) $(T_J(x), T_K(y)) = (T_{J\cap K}(x), T_{J\cap K}(y)) = |J \cap K|(x, y)$, *where* $x, y \in R^\perp$, J, K *are intervals and* $|\ |$ *is the length.*

From (4.5)(c) we see at once that

(4.6) $\mathfrak{N}_{ab}$ *is a (closed) subspace of* $\mathfrak{X}$, *and* $\dim.\mathfrak{N}_{ab}=\dim.R^{\perp}$, $0\leqq a<b$.

But it should be noted that our subspace-valued measure $\mathfrak{N}_{ab}$ is only *subadditive*, i.e. $\mathfrak{N}_{ac}\subset\mathfrak{N}_{ab}+\mathfrak{N}_{bc}$, $0\leqq a<b<c$; for, we find that

$$(4.7)\qquad \mathfrak{N}_{ac}^{\perp}\cap(\mathfrak{N}_{ab}+\mathfrak{N}_{bc})=\left(\frac{1}{b-a}T_{ab}-\frac{1}{c-b}T_{bc}\right)(R^{\perp}),$$

and the last is not $\{0\}$ even when $\dim.\ R^{\perp}=1$.

A simple but important consequence of (4.2)(b) and (4.3) is the identity

$$(4.8)\qquad \mathfrak{S}\{S_t(R^{\perp})\}_{t\geqq a}=\mathfrak{S}(\mathfrak{N}_{st})_{a\leqq s<t<\infty}=\mathfrak{S}\{T_{st}(R^{\perp})\}_{a\leqq s<t<\infty}.$$

This identity enables us to restate the pre-Wold decomposition (3.3) in the form

$$(4.9)\qquad \mathfrak{X}_a=\mathfrak{S}\{T_{st}(R^{\perp})\}_{a\leqq s<t<\infty}+\mathfrak{X}_\infty,\qquad (a\geqq 0),\ T_{st}(R^{\perp})\perp\mathfrak{X}_\infty.$$

On comparing this with the corresponding decomposition in the discrete case (cf. (1.1), (3.1)), viz.

$$\mathfrak{X}_n=\mathfrak{S}\{V^k(R^{\perp})\}_{k\geqq n}+\mathfrak{X}'_\infty,\ (n\geqq 0),\qquad V^k(R^{\perp})\perp\mathfrak{X}'_\infty,$$

we see that the subspaces $T_{st}(R^{\perp})$ have taken the place of the "innovation subspaces" $V^k(R^{\perp})$. This fact along with the properties (4.5)(b), (4.6) justifies our calling $T_{st}(R^{\perp})$, $0\leqq s<t$, the *differential innovation subspaces of* $\mathfrak{X}$ *engendered by the semi-group* $(S_t,\ t\geqq 0)$.

Now in the discrete case we have the *direct sum* representation:

$$\mathfrak{S}\{V^k(R^{\perp})\}_{k\geqq 0}=\sum_{k=0}^{\infty}V^k(R^{\perp}),$$

where, by definition,

$$\sum_{k=0}^{\infty}V^k(R^{\perp})=\left\{\xi:\ \xi=\sum_{k=0}^{\infty}V^k(x_k),\ x_k\in R^{\perp}\ \&\ \sum_{k=0}^{\infty}|x_k|^2<\infty\right\}.$$

This suggests that in the continuous case we should have an analogous *direct integral* representation:

$$\mathfrak{S}\{(T_{st}(R^{\perp})\}_{0\leqq s<t<\infty}=\int_0^{\infty}T_{dt}(R^{\perp}),$$

where

$$\int_0^{\infty}T_{dt}(R^{\perp})=\left\{\xi:\ \xi=\int_0^{\infty}T_{dt}(x_t),\ x_t\in R^{\perp}\ \&\ \int_0^{\infty}|x_t|^2dt<\infty\right\}.$$

This heuristic reasoning can be put on a sound footing by defining precisely the vector-valued integral $\int_0^\infty T_{dt}(x_t)$ occurring in the last equation. This is done in §§5, 6 below.

5. **Generalized vector-valued integrals.** Let $L_2([a, b], R^\perp)$ be the Hilbert space of all strongly (Lebesgue) measurable functions x on $[a, b]$ with values $x_t \in R^\perp$ such that $\int_a^b |x_t|^2 dt < \infty$.[6] Our task is to define $\int_a^b T_{dt}(x_t)$ so that it will behave like a vector sum $\sum_{k=m}^n V^k(x_k)$, where $x_k \in R^\perp$. This suggests that we define it so as to ensure the following properties: for all functions $x, y, x^{(n)} \in L_2([a, b], R^\perp)$,

$$
(5.1)\quad
\begin{aligned}
&\text{(a)} \left(\int_a^b T_{dt}(x_t), \int_a^b T_{dt}(y_t)\right) = \int_a^b (x_t, y_t)dt,\\
&\text{(b)} \left|\int_a^b T_{dt}(x_t)\right|^2 = \int_a^b |x_t|^2 dt,\\
&\text{(c)} \int_a^b T_{dt}(cx_t + dy_t) = c\int_a^b T_{dt}(x_t) + d\int_a^b T_{dt}(y_t),\\
&\text{(d)} \int_a^b T_{dt}(x_t^{(n)}) \to \int_a^b T_{dt}(x_t), \textit{ when } x^{(n)} \to x \textit{ in the } L_2\textit{-topology.}
\end{aligned}
$$

The requisite definition consists of two parts, one for step-functions x and the other for arbitrary x in $L_2([a, b], R^\perp)$:

5.2(a). Definition. *For the step-function* $x = \sum_{k=1}^n \alpha_k \chi_{J_k}$ *on* $[a, b]$, *where* $\alpha_k \in R^\perp$ *and* χ_{J_k} *is the indicator-function of the bounded interval* J_k *we define* $\int_a^b T_{dt}(x_t) = \sum_{k=1}^n T_{J_k}(\alpha_k)$.

It follows from (4.5) that this definition is unequivocal and that the laws (5.1)(a)–(c) hold when x and y are step-functions. Moreover, for any Cauchy-sequence of step-functions $x^{(n)}$ in $L_2([a, b], R^\perp)$ we have

$$\left|\int_a^b T_{dt}(x_t^{(m)}) - \int_a^b T_{dt}(x_t^{(n)})\right|^2 = \int_a^b \left|x_t^{(m)} - x_t^{(n)}\right|^2 dt \to 0,$$

as $m, n \to \infty$. This relation and the well-known fact that the step-functions are everywhere dense in $L_2([a, b], R^\perp)$ suggest the following extension of our definition:

5.2(b). Definition. *For any* $x \in L_2([a, b], R^\perp)$, *we define* $\int_a^b T_{dt}(x_t) = \lim_{n\to\infty} \int_a^b T_{dt}(x_t^{(n)})$, *where* $(x^{(n)}, n \geq 1)$ *is any sequence of step-functions tending to* x *in the* L_2*-topology.*

It is easy to check that our definition is again unequivocal, and

[6] Cf. [**3**, Chapter III, §6]. According to their Theorem 6, $L_2([a, b], R^\perp)$ is a Banach space. With the inner product $(x, y) = \int_a^b (x_t, y_t)dt$, it is obviously a Hilbert space.

that the laws (5.1) hold without restriction. Moreover, as an interval-function the integral is seen to have the following properties:

$$(5.3)\quad \begin{aligned} &\text{(a)}\ \int_a^b T_{dt}(x_t) + \int_b^c T_{dt}(x_t) = \int_a^c T_{dt}(x_t), \qquad 0 \leqq a < b < c, \\ &\text{(b)}\ \int_J T_{dt}(x_t) \perp \int_K T_{dt}(y_t), \qquad J,\ K \textit{ non overlapping}, \\ &\text{(c)}\ \left(\int_J T_{dt}(x_t),\ \int_K T_{dt}(y_t)\right) = \int_{J\cap K} (x_t, y_t)dt, \\ &\text{(d)}\ S_c\left\{\int_a^b T_{dt}(x_t)\right\} = \int_{a+c}^{b+c} T_{ds}(x_{s-c}). \end{aligned}$$

From (5.1) and (5.3) we see that our vector-valued integral has properties akin to those possessed by stochastic integrals.[7] To see the precise relationship between the two concepts, consider the function $x_t = c(t)\alpha$, where $\alpha \in R^\perp$ and $c(\cdot)$ is a complex-valued function in $L_2[a, b]$, and let $\xi_t = T_t(\alpha)$. Then it follows easily that the process $(\xi_t,\ t \geqq 0)$ has orthogonal increments, and

$$(5.5)\qquad \int_a^b T_{dt}\{c(t)\alpha\} = \int_a^b c(t)d\xi_t \quad \textit{(stochastic integral)}.$$

This shows that *our notion of vector-integration subsumes that of stochastic integration, but reduces to the latter when and only when* dim. $R^\perp = 1$.

6. The direct integral. We can now define our direct integral as a set of vector-valued integrals:

$$(6.1)\qquad \int_a^b T_{dt}(R^\perp) = \left\{\xi \colon \xi = \int_a^b T_{dt}(x_t),\ x \in L_2([a, b], R^\perp)\right\},$$

where $0 \leqq a < b$. By (5.1)(c), (d) this integral is a (closed) subspace of X. Indeed, (5.1) enables us at once to assert the following theorem:

6.2. THEOREM. *The correspondence $x \to \int_a^b T_{dt}(x_t)$ is an isomorphism on the Hilbert space $L_2([a, b], R^\perp)$ onto the subspace $\int_a^b T_{dt}(R^\perp)$ of $\mathfrak{X}$, $0 \leqq a < b$.*

From (5.3) we see, moreover, that as an interval-function our

[7] Such integrals were introduced in probability theory by Wiener, Cramer and Doob. They also occur in Hilbert space theory when spectral integrals $\int_a^b c(\lambda)dE_\lambda$, where $(E_\lambda,\ a \leqq \lambda \leqq b)$ is a resolution of I, are applied to vectors. Cf. [2, Chapter IX, §2], and [9, Chapter VI, §2].

direct integral has the following convenient properties:

$$(6.3)\quad \begin{aligned}
&\text{(a)} \int_a^b T_{dt}(R^\perp) + \int_b^c T_{dt}(R^\perp) = \int_a^c T_{dt}(R^\perp), \quad 0 \leqq a < b < c,\\
&\text{(b)} \int_J T_{dt}(R^\perp) \perp \int_K T_{dt}(R^\perp), \quad J, \quad K \textit{ nonoverlapping},\\
&\text{(c)} \int_J T_{dt}(R^\perp) \subseteq \int_K T_{dt}(R^\perp), \qquad J \subseteq K,\\
&\text{(d)} S_c\left\{\int_a^b T_{dt}(R^\perp)\right\} = \int_{a+c}^{b+c} T_{dt}(R^\perp).
\end{aligned}$$

We can also show that

$$(6.4)\qquad \int_a^b T_{dt}(R^\perp) = \mathfrak{S}\{T_{\sigma\tau}(R^\perp)\}_{a \leq \sigma < \tau < b}.$$

This relation with $b = \infty$ together with (4.9) yields the result we had set out to prove:

6.5. THEOREM (WOLD DECOMPOSITION). *Let $(S_t, t \geqq 0)$ be a strongly continuous semi-group of isometries on $\mathfrak{X}$ into $\mathfrak{X}$, iH be its infinitesimal generator and V the Cayley transform of H. Then for $a \geqq 0$*

$$S_a(\mathfrak{X}) = \int_a^\infty T_{dt}(R^\perp) + \mathfrak{X}_\infty, \qquad \int_0^\infty T_{dt}(R^\perp) \perp \mathfrak{X}_\infty,$$

where $R = V(\mathfrak{X})$ and $\mathfrak{X}_\infty = \bigcap_{t \geq 0} S_t(\mathfrak{X})$.

From this decomposition we can readily obtain Cooper's theorem that our semi-group can be embedded in a unitary group acting on a larger Hilbert space [1, p. 841].

Our direct integral does not bear any obvious relation to the direct integral $\int_a^b \mathfrak{K}_t d\mu(t)$ due to von Neumann and others, cf. [7], in which $\mathfrak{K}_t$ is a Hilbert space and μ a complex-valued measure. Our integral could be written in the form $\int_a^b d\mathfrak{N}_t$, on letting $\mathfrak{N}_t = T_{0t}(R^\perp)$, cf. (4.4). But the significant factor in its definition is the family of operators T_{0t} and not the family of subspaces $\mathfrak{N}_t$, cf. Definitions 5.2(a), (6.1). It would seem that this integral is the tool needed for the study of the isometric representations of continuous semi-groups, just as the von Neumann integral is the tool required to deal with the unitary representations of continuous groups.

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