GROUPS WITH INFINITE PRODUCTS

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Communicated by Deane Montgomery, April 8, 1962

If G is a group, then an *infinite product* on G is a function $\mu: G^\infty \to H$, where G^∞ is the set of sequences of elements of G, and H is some other set. I call μ *associative* if it satisfies all the associative laws of the form

$$\begin{aligned} &\mu(x_1, x_2, \cdots, x_n, \cdots) \\ &\quad = \mu(x_1 x_2 \cdots x_{i_2-1}, x_{i_2} \cdots x_{i_3-1}, \cdots, x_{i_n} \cdots x_{i_{n+1}-1}, \cdots). \end{aligned}$$

Here juxtaposition denotes multiplication in G. Then the utter triviality of μ follows from this trick:

$$\begin{aligned} &x_1 x_2 x_3 \cdots \\ &\quad = (x_1 \bar{x}_1 x_1)(x_2 \bar{x}_2 \bar{x}_1 x_1 x_2)(x_3 \bar{x}_3 \bar{x}_2 \bar{x}_1 x_1 x_2 x_3) \cdots \\ &\quad = (x_1 \bar{x}_1)(x_1 x_2 \bar{x}_2 \bar{x}_1)(x_1 x_2 x_3 \bar{x}_3 \bar{x}_2 \bar{x}_1) \cdots \\ &\quad = 1 \cdot 1 \cdot 1 \cdots . \end{aligned}$$

Here $\bar{x}_n$ denotes the inverse of x_n, and 1 denotes the identity of G.

A form of this trick was noticed and used by B. Mazur [**2**]. An example of another use is this.

Let C be a compact Hausdorff space. If $\alpha > 0$, define $C(\alpha)$ to be the space $C \times [0, \alpha)$ with $C \times 0$ identified to one point 0. Define Σ to be the set of all those functions $f: C(1) \to C(1)$ which can be extended to $f_*: C(2) \to C(1)$ where f_* is a homeomorphism onto an open subset of $C(1)$, such that $f(0) = 0$. Define Γ to be the set of those homeomorphisms $\phi: C(1) \to C(1)$ for which there is $\epsilon > 0$ such that ϕ is the identity on $C(\epsilon) \cup [C(1) - C(1-\epsilon)]$.

If f and g belong to Σ, define $f \sim g$ to mean there exists $\phi \in \Gamma$ such that $f = g\phi$, where the notation here for composition of maps is that $g\phi(x) = \phi(g(x))$. It can be shown that $f \sim g$ if and only if there is $\epsilon > 0$ such that $f | C(\epsilon) = g | C(\epsilon)$.

From this, one can deduce that the equivalence classes of Σ under the relation $\sim$ form a group with multiplication induced by the composition of maps; this group will be called G.

Now if $f_1, f_2, \cdots$, is a sequence of elements of Σ, define $\mu(f_1, f_2, \cdots)$ to be the direct limit of the sequence of spaces and maps

$$C(1) \xrightarrow{f_1} C(1) \xrightarrow{f_2} C(1) \to \cdots .$$

[1] This research was supported by the Air Force Office of Scientific Research.

One can show that $\mu(f_1, f_2, \cdots)$ is determined up to homeomorphism by the equivalence classes of f_1, f_2, etc. The associativity, up to homeomorphism, of μ is simply the statement that the direct limit of a directed set of spaces and maps is homeomorphic to the direct limit of a cofinal subset.

The associativity trick then proves that the spaces $\mu(f_1, f_2, \cdots)$ are all homeomorphic to each other; a particular such space can be shown homeomorphic to $C(\infty)$ or $C(1)$.

It follows from the compactness of C, that if X is a space which is the union of its open subsets U_n, each of which is homeomorphic to $C(\infty)$ in such a way that the odd points 0 coincide for all n, and if every compact subset of X is contained in some U_n, then X is homeomorphic to some space of the form $\mu(f_1, f_2, \cdots)$. And hence X is homeomorphic to $C(\infty)$.

Taking C to be the $(n-1)$-sphere, one obtains the theorem of M. Brown [1] that a monotone union of open n-cells is an open n-cell.

This is perhaps the most conceptual way to understand my proof [3] of several generalizations of Brown's theorem, although if written out in detail this method would be no shorter.

References

1. M. Brown, *The monotone union of open n-cells is an open n-cell*, Proc. Amer. Math. Soc. **12** (1961), 812–814.

2. B. Mazur, *On embeddings of spheres*, Bull. Amer. Math. Soc. **65** (1959), 59–65.

3. J. Stallings, *On a theorem of Brown about the union of open cones*, Ann. of Math. (to appear).

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