RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

INVARIANT QUADRATIC DIFFERENTIALS[1]

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Communicated by R. Bott, March 21, 1962

Let S be a compact Riemann surface of genus $g \geqq 2$ and h an automorphism (conformal homeomorphism onto itself) of S. h generates a cyclic group $H = \{I, h, \cdots, h^{N-1}\}$ where N is the order of h. We shall assume that N is a prime number. Let D_m for an integer $m \geqq 0$ denote the space of meromorphic differentials on S and $A_m \subset D_m$ the subspace of finite analytic (without poles) differentials. We obtain representations of H by assigning to h the linear transformation of D_m into itself by $h(\theta) = \theta h^{-1}$ for every $\theta \in D_m$. It is clear that h takes A_m into itself so that by restricting to A_m we have a representation of H by a group of linear transformations of a finite dimensional vector space.

In this note we are concerned with determining some of the properties of (h), the diagonal matrix for h, considering h as a linear transformation on the $3g-3$ dimensional space A_2 of quadratic differentials. Since $(h)^N = (I)$ it is clear that each diagonal element of (h) is an Nth root of unity. If $\epsilon \neq 1$ is an Nth root of unity, denote by n_k the multiplicity of ϵ^k $(k = 0, 1, \cdots, N-1)$ in (h).

Let $\hat{S} = S/H$ be the orbit space of S under H. Then it is well known that $\hat{S}$ can be given a conformal structure and the projection map $\pi: S \to \hat{S}$ is then analytic. The branch points of this covering are precisely at the t fixed points of h, $P_1, \cdots, P_t \in S$, $t \geqq 0$—here we make essential use of the assumption that N prime—each a branch point of order $N-1$. Let g_1 be the genus of $\hat{S}$. The Riemann-Hurwitz formula reads $2g-2 = N(2g_1-2)+(N-1)t$. Now clearly n_0 is the dimension of that subspace of A_2 which consists of H—invariant differentials, i.e., those satisfying $h(\theta) = \theta$.

THEOREM 1. (i) *n_0, the dimension of the space of H-invariant finite quadratic differentials, is $3g_1 - 3 + t$.*

(ii) *If $n_k \neq 0$ for some k, $1 \leqq k \leqq N-1$, then*

[1] This is a brief edited excerpt from my thesis submitted to Yeshiva University, 1962.

$$(*)\qquad 3g_1 - 3 + 2\frac{(N-1)}{N}t \geqq n_k \geqq 3g_1 - 3 + \frac{(N-1)}{N}t.$$

(iii) *There exists* k^*, $1 \leqq k^* \leqq N-1$, *for which* $n_{k^*} \neq 0$.

(iv) *If* $g_1 \geqq 1$ *then* $n_0 \leqq 3g-5$ *unless* S *is a surface with equation* $y^2 = x^6 + Ax^4 + Bx^2 + 1$, *in which case* $g=2$, $g_1 = 1$, $n_0 = 2 = 3g-4 = 3g_1 - 3 + t$.

The proof of (i) is similar to the proof of (ii) given below. (iii) follows immediately from the

LEMMA. *The representation* $h^m \to (h)^m$, $m = 0, 1, \cdots N-1$, *of* H *is faithful, i.e.*, $(h)^m = (I)$ *implies* $m=0$ *unless* $g=2$ *and* $h=J$, *the hyperelliptic involution.*

The simple proof of this lemma is in my thesis and is omitted here.

(iv) is an immediate consequence of (iii) and (ii) since (*) then implies $n_{k^*} \geqq 2$ unless $g_1 = 1$, $N=2$, $t=2$ ($g_1=1$ implies $t \geqq 1$ by Riemann-Hurwitz) or $g_1 = 1$, $t=1$. But if $g_1=1$, $t=1$ then by (i) one has $n_0 = 1 \leqq 3g-5$ for $g \geqq 2$. The first exception is the case indicated in (iv) with $h: x \to -x$, $y \to y$ and fixed points on the two sheets over $x=0$.

To prove (ii) let $\theta \in A_2$ be such that $h(\theta) = \epsilon^k\theta$. At any fixed point $P \in \{P_1 \cdots P_t\}$ say $h^{-1}: z \to \eta z$ in terms of a suitable local parameter, $\eta^N = 1$, $\eta \neq 1$. Then we must have $\theta h^{-1} = (a_0 + a_1(\eta z) + \cdots)\eta^2 dz^2 = \epsilon^k(a_0 + a_1 z + \cdots)dz^2$. Thus $a_n = 0$ unless $n+2 \equiv l \pmod N$ where $\eta^l = \epsilon^k$; $1 \leqq l \leqq N-1$. θ then actually has an expansion of the form at P in z,

$$\theta = (a_{l-2}z^{l-2} + \cdots + a_{kN+l-2}z^{kN+l-2} + \cdots)dz^2$$

(if $l \geqq 2$; if $l=1$ the first term must be omitted). This then holds for every θ for which $h(\theta) = \epsilon^k\theta$. To each point P_i, $i = 1 \cdots t$, we have then $\eta_i^{l_i} = \epsilon^k$, for suitable η_i, l_i. Such a θ then necessarily has at P_i a zero of the form $r_iN + l_i - 2 \geqq 0$ and the divisor of θ must be $(\theta) = (P_i^{r_iN+l_i-1}Q_j^{m_j}h(Q_j)^{m_j} \cdots h^{N-1}(Q_j)^{m_j})$ where the Q_j are nonfixed points of h.

Let us partition the P_i into $P_1 \cdots P_u$, and $P_{u+1} \cdots P_t$, $0 \leqq u \leqq t$, where P_i for $i \leqq u$ has $l_i = 1$ and P_i for $i > u$ has $l_i \geqq 2$. If $h(\phi) = \epsilon^k\phi$ also, then $\phi/\theta = f$ is an H invariant function on S which may be construed as a function $\hat{f}$ on $\hat{S}$. Then, since $f\theta$, for fixed θ and f varying over all H invariant functions with poles at most at the zeros of θ, gives us all differentials $\phi \in A_2$ for which $h(\phi) = \epsilon^k\phi$, we have to compute the dimension of this space of functions on $\hat{S}$. At a point P_i, $i \leqq u$, $\phi/\theta = f$ is

$$\frac{z^{r'_iN-1}+\cdots}{z^{r_iN-1}+\cdots}=z^{-(r_i-r'_i)N\theta}+\cdots$$

but r_i' is at least 1, so that f has a pole of order at most $(r_i-1)N$. On the other hand, at P_i, $i>u$, $\phi/\theta=f$ is $z^{-(r_i-r'_i)N}$ where r_i' may be 0, so that f may have a pole of order at most r_iN. Thus, on $\hat{S}$, $\hat{f}$ must be a multiple of the divisor

$$\omega=(\hat{P}_1^{1-r_1}\cdots\hat{P}_u^{1-r_u}\hat{P}_{u+1}^{-r_{u+1}}\cdots\hat{P}_t^{-r_t}\hat{Q}_j^{-m_j}).$$

We now have $n_k=\deg\ (\omega^{-1})+i(\omega^{-1})+1-g_1$. The degree of the divisor (θ) is

$$\begin{aligned}4g-4&=\sum_{i=1}^{t}(r_iN+l_i-2)+N\sum m_j\\&=N\left(\sum_{i=1}^{t}r_i+\sum m_j\right)+\sum_{i=u+1}^{t}(l_i-2)-u.\end{aligned}$$

Therefore,

$$\deg(\omega^{-1})=\sum_{i=1}^{t}r_i-u+\sum m_j=\frac{4g-4-\sum_{i=u+1}^{t}(l_i-2)-(N-1)u}{N}.$$

This is as small as possible when $u=t$ and as large as possible when $u=0$ and each $l_i=2$. When $u=t$ we have $\deg\ (\omega^{-1})=(4g-4)/N-((N-1)/N)t$. Using the Riemann-Hurwitz relation gives, $\deg\ (\omega^{-1})=4g_1-4+((N-1)/N)t>2g_1-2$, so that $i(\omega^{-1})=0$ in any event. When $u=0$ and each $l_i=2$, we have $\deg\ (\omega^{-1})=(4g-4)/N=4g_1-4+2((N-1)/N)t$. This completes the proof of (ii).

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