A NON-HOPFIAN GROUP[1]

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The purpose of this note is to construct a non-hopfian[2] and finitely generated group S which is in no way complicated. This group S is a generalised free square[3] of the free nilpotent group A of class two on two generators. Since A satisfies the maximum condition, S differs radically from the non-hopfian groups constructed by Graham Higman [**1**], with which it may be compared. It may perhaps be of interest to point out that the first finitely generated non-hopfian groups were constructed by B. H. Neumann [**2**]. Besides these and the non-hopfian groups of Higman [**1**], the only other known finitely generated non-hopfian groups are those due to B. H. Neumann and Hanna Neumann [**3**] and to P. Hall [**4**].

The construction of S is as follows. We take a replica B of A. Thus we may present A and B as follows:

$$A = gp(a, b; [a, b, b] = [a, b, a] = 1),$$
$$B = gp(c, d; [c, d, d] = [c, d, c] = 1).$$

Here, as is the custom, we define

$$[x, y] = x^{-1}y^{-1}xy, \qquad [x, y, z] = [[x, y], z], \qquad x^y = y^{-1}xy,$$

where x, y, z belong to some group G.

We now define

$$H = gp(a, [a^2, b]) \quad \text{and} \quad K = gp([c, d], c).$$

It is easy to verify that H and K are free abelian of rank two and hence isomorphic. Therefore we can form the generalised free product S of A and B amalgamating H with K:

$$S = (A * B; a = [c, d], [a^2, b] = c).$$

It is clear that we may present S as follows:

$$S = gp(a, b, d; [a, b, b] = [a, b, a] = 1,$$
$$[[a^2, b], d, d] = [[a^2, b], d, [a^2, b]] = 1, a = [a^2, b, d]).$$

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[2] A group G is hopfian if it is not isomorphic to any proper factor group of itself. If G is not hopfian, we say G is non-hopfian.

[3] A generalised free square is, by definition, the generalised free product of two isomorphic groups.

Since $[a, [a^2, b]] = 1$ is a consequence of

$$[a, b, a] = 1 \quad \text{and} \quad [a, b, b] = 1,$$

we have

$$S = gp(a, b, d; [a, b, b] = [a, b, a] = [a, d] = 1, a = [a^2, b, d]). \tag{1}$$

We observe next that a has a square root $\tilde{a}$ in S, where

$$\tilde{a} = ([a, b])^{db^{-2}}. \tag{2}$$

For

$$\begin{aligned} \tilde{a}^2 &= ([a, b]^{db^{-2}})^2 = ([a, b]^2)^{db^{-2}} = [a^2, b]^{db^{-2}} = ([a^2, b][a^2, b, d])^{b^{-2}} \\ &= (a[a^2, b])^{b^{-2}} = a[a, b^{-2}][a^2, b] = a[a, b]^{-2}[a, b]^2 = a. \end{aligned}$$

The existence of this square root $\tilde{a}$ of a enables us to present S in a slightly different way. To this end, let

$$\begin{aligned} S' = gp(a', b', d'; [a'^2, b', a'^2] &= [a'^2, b', b'] = [a'^2, d'] = 1, \\ a'^2 &= [a'^4, b', d'], \ a' = [a'^2, b']^{d'b'^{-2}}). \end{aligned} \tag{3}$$

We now define a mapping θ of the given generators of S' into S as follows:

$$a'\theta = \tilde{a}, \qquad b'\theta = b, \qquad d'\theta = d.$$

It is easy to check that the images of a', b' and d' under θ satisfy the relations corresponding to the relations satisfied by a', b' and d' (cf. (1), (3) and (2)). So, by Dyck's theorem (cf. e.g. Kurosh [5, vol. 1, p. 130]) θ can be extended to a homomorphism θ' of S' onto S.

On the other hand, we define a mapping ϕ of the generators of S into S':

$$a\phi = a'^2, \qquad b\phi = b', \qquad d\phi = d'.$$

Again an application of Dyck's theorem is permissible. So ϕ can be extended to a homomorphism ϕ' of S into S'.

Now it is easy to see (cf. (2)) that

$$\theta'\phi' = 1,$$

the identity automorphism of S'. So θ' is an isomorphism from S' onto S.

Let us define, finally, a second mapping ψ of the generators of S' into S:

$$a'\psi = a, \qquad b'\psi = b, \qquad d'\psi = d.$$

Once more an application of Dyck's theorem is in order, since again the images of a', b' and d' under ψ can be shown to satisfy, after a brief calculation, the corresponding relations (cf. (3) and (1)). So ψ too can be extended to an epimorphism ψ' of S' to S. The kernel K of ψ' clearly contains

$$w' = [a', b', b'].$$

The element $w' \neq 1$; to see this it is enough to check that

$$w'\theta' = [\tilde{a}, b, b] \neq 1.$$

This last check is made easy because S is the generalised free product of A and B amalgamating H with K. All we do is expand $w'\theta'$ into a product of $a, a^{-1}, b, b^{-1}, d, d^{-1}$ and then check that the product can be broken up into a product of elements coming alternately out of A and B, but not out of both; this last fact is a sufficient condition for $w'\theta' \neq 1$ (cf. B. H. Neumann [6, p. 511]). It follows, therefore, that K is nontrivial.

We have then a chain of isomorphisms:

$$S' \cong S \cong S'/K.$$

So S' (and hence also S) is isomorphic to a proper factor group of itself. Thus we have proved the following theorem.

THEOREM. *Let A be the free nilpotent group of class two on two generators. Then there exists a generalised free square of A which is non-hopfian.*

To end with we mention that the generalised free product often leads to groups that *are* hopfian—this occurs if we restrict the amalgamation to be cyclic, for example (cf. G. Baumslag [7; 8]).

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