RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

SUPPORTS OF A CONVEX FUNCTION[1]

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Let C be a real, symmetric, $m\times m$, positive-semi-definite matrix. Let $R^m = \{(x_1, \cdots, x_m) \mid x_i \text{ is a real number}, i=1, \cdots, m\}$, and let $K \subset R^m$ be a polyhedral convex cone, i.e., there exists a real $m\times n$ matrix A such that $K = \{x \mid x\in R^m \text{ and } xA \leqq 0\}$. Consider the function $\psi: K\to R$ defined by $\psi(x) = (xCx^T)^{1/2}$ for all $x\in K$. We wish to characterize the set, U, of all supports of ψ, where

$$(1) \qquad U = R^m \cap \{u \mid x \in K \Rightarrow ux^T \leqq (xCx^T)^{1/2}\}.$$

Let $R^n_+ = R^n \cap \{\pi \mid \pi \geqq 0\}$ and consider the set

$$(2)\ V = \{v \mid \exists x \in R^m, \pi \in R^n_+ \text{ and } v = \pi A^T + xC, xCx^T \leqq 1, \quad xA \leqq 0\}.$$

We shall demonstrate:

THEOREM. $U = V$.

We first show:

LEMMA 1. $x, y\in R^m \Rightarrow (xCy^T)^2 \leqq (xCx^T)(yCy^T)$.

PROOF. If $x, y\in R^m$ consider the polynomial $p(\lambda) = \lambda^2 xCx^T + 2\lambda xCy^T + yCy^T = (x+\lambda y)C(x+\lambda y)^T$. Since C is positive-semi-definite, $p(\lambda) \geqq 0$ for all real numbers λ, and thus the discriminant of p is nonpositive, i.e.,

$$4(xCy^T)^2 - 4(xCx^T)(yCy^T) \leqq 0. \qquad \text{q.e.d.}$$

As an immediate application of Lemma 1 we show:

LEMMA 2. $V \subset U$.

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PROOF. Let $v \in V$, then there exist $x \in R^m$, $\pi \in R_+^n$ such that $v = \pi A^T + xC$, $xCx^T \leqq 1$. Now if $y \in R^m$, $yA \leqq 0$, then $vy^T = yA\pi^T + xCy^T$ and $vy^T \leqq xCy^T$, because $yA \leqq 0$, $\pi^T \geqq 0$ and $yA\pi^T \leqq 0$. Thus, $vy^T \leqq (xCx^T)^{1/2}(yCy^T)^{1/2}$, by Lemma 1, and $vy^T \leqq (yCy^T)^{1/2}$, because $xCx^T \leqq 1$. Thus, $v \in U$. q.e.d.

From the fact that C is positive-semi-definite, it follows that:

LEMMA 3. *The set V is convex.*

PROOF. If $x_k \in R^m$, $\pi_k \in R_+^n$, $x_kA \leqq 0$, $u_k = \pi_k A^T + x_kC$, $x_kCx_k^T \leqq 1$, $\lambda_k \in R_+$ for $k=1, 2$ and $\lambda_1 + \lambda_2 = 1$, then: $\lambda_1 u_1 + \lambda_2 u_2 = (\lambda_1\pi_1 + \lambda_2\pi_2)A^T + (\lambda_1 x_1 + \lambda_2 x_2)C$, $(\lambda_1 x_1 + \lambda_2 x_2)A \leqq 0$, $\lambda_1 x_1 + \lambda_2 x_2 \in R^m$, $\lambda_1\pi_1 + \lambda_2\pi_2 \in R_+^n$, and

$$
\begin{aligned}
(\lambda_1 x_1 + \lambda_2 x_2)C(\lambda_1 x_1 + \lambda_2 x_2)^T - 1 \\
&\leqq (\lambda_1 x_1 + \lambda_2 x_2)C(\lambda_1 x_1 + \lambda_2 x_2)^T - \lambda_1 x_1 C x_1^T - \lambda_2 x_2 C x_2^T \\
&= -\lambda_1\lambda_2[x_1Cx_1^T - 2x_1Cx_2^T + x_2Cx_2^T] \\
&= -\lambda_1\lambda_2(x_1 - x_2)C(x_1 - x_2)^T \leqq 0,
\end{aligned}
$$

because C is positive-semi-definite. q.e.d.

LEMMA 4. *The set V is closed.*

PROOF. Let $\{w_k\}$ be a sequence with $w_k \in R^m$, $k = 1, 2, \cdots$. We define the (pseudo) norm of w_k, denoted $|\{w_k\}|$, to be the smallest non-negative integer p such that there exists a k_0 and for all $k \geqq k_0$, x_k has at most p nonzero components. Now, suppose u is in the closure of V, i.e., there exist sequences $\{u_k\}$, $\{\pi_k\}$ and $\{x_k\}$ such that

$$
\tag{3}
\begin{aligned}
&\pi_k \in R_+^n, \quad x_k \in R^m, \qquad u_k = \pi_k A^T + x_k C, \\
&x_k A \leqq 0 \quad \text{and} \quad y_k C x_k^T \leqq 1, \qquad\qquad k = 1, 2, \cdots \\
&\text{and } \{u_k\} \text{ converges to } u.
\end{aligned}
$$

Suppose the sequence $\{x_k\}$ is bounded, then we may assume, having taken an appropriate subsequence, that for some $x \in R^m$, $\{x_k\} \to x$ and thus, by (3), $xA \leqq 0$ and $xCx^T \leqq 1$. Now, $yA \leqq 0 \Rightarrow u_k y^T - x_k C y^T = \pi_k A^T y^T = yA\pi_k^T \leqq 0$, all $k \Rightarrow uy^T - xCy^T \leqq 0$. Thus the system,

$$
\begin{aligned}
y &\in R^m, \\
yA &\leqq 0, \\
(u - xC)y^T &> 0,
\end{aligned}
$$

has no solution and by the usual feasibility theorem for linear inequalities (see e.g. [4] or [5]) the system:

$$\pi \in R^n_+,$$
$$\pi A^T = u - xC,$$

has a solution, and thus $u \in V$.

We have just demonstrated that if $\{x_k\}$ is bounded, then $u \in V$. Since $|\{x_k\}| + |\{x_kA\}| \leqq m+n$, it is always possible to choose $\{x_k\}$ and $\{\pi_k\}$ satisfying (3) and such that $|\{x_k\}| + |\{x_kA\}|$ is minimal. We shall show next that if $\{x_k\}$, $\{\pi_k\}$ are so chosen, then $\{x_k\}$ is indeed bounded, thus completing the proof. Suppose then that $\{x_k\}$ is not bounded, i.e., $\exists$ a subsequence such that $|x_k| = (x_kx_k^T)^{1/2} \to \infty$, and we may assume that $|x_k| > 0$ for all k. Let

$$z_k = \frac{x_k}{|x_k|}, \qquad k = 1, 2, \cdots,$$

then $\{z_k\}$ is bounded and we may assume that there is a $z \in R^m$ such that the z_k converge to z and $|z| = 1$. From (3) it follows that $z_kA \leqq 0$ and $z_kCz_k^T \leqq 1/|x_k|$ for all k. Thus, $zA \leqq 0$ and $zCz^T \leqq 0$. But then, from Lemma 1, $zCy^T = 0$ for all $y \in R^m$, and $zC = 0$. Summarizing:

$$z \in R^m, \qquad zA \leqq 0, \qquad zC = 0. \tag{4}$$

Note that if z has a nonzero component, then infinitely many x_k's must have the same component nonzero, this follows from the fact that z is the limit of $x_k/|x_k|$. As a consequence, if $\{\lambda_k\}$ is any sequence of real numbers, then $|\{x_k+\lambda_kz\}| \leqq |\{x_k\}|$. If $zA \neq 0$, and a^j, $j=1, \cdots, n$, denotes the jth column of A, let

$$\lambda_k = \max \left\{ \frac{z_ka^j}{za^j} \; j = 1, \cdots, n \text{ and } za^j < 0 \right\}.$$

Then we may replace, in (3), x_k by $x_k + \lambda_k z$ because $\lambda_k za^j + x_ka^j \leqq 0$ for all j, and $(x_k+\lambda_kz)A \leqq 0$, also $zC = 0$ and thus $(x_k+\lambda_kz)C = x_kC$, $(x_k+\lambda_kz)C(x_k+\lambda_kz)^T = x_kCx_k^T \leqq 1$. However each $(x_k+\lambda_kz)A$ has at least one more zero component than x_kA, contradicting the minimality of $|\{x_k\}| + |\{x_kA\}|$. Thus, $zA = 0$ and we may replace, in (3), x_k by $x_k+\lambda_kz$ for an *arbitrary sequence* $\{\lambda_k\}$. But $z \neq 0$ and we can define λ_k so that $x_k+\lambda_kz$ has at least one more zero component than x_k has, thus $|\{x_k+\lambda_kz\}| < |\{x_k\}|$. However, $(x_k+\lambda_kz)A = x_kA$, and $|\{(x_k+\lambda_kz)A\}| = |\{x_kA\}|$, contradicting the minimality assumption. q.e.d.

Lastly, we show:

LEMMA 5. *$U \subset V$.*

PROOF. Suppose $u \notin V$. By Lemmas 3 and 4 V is a closed convex set, hence there is a hyperplane which separates u *strongly* from V (see [4]). Thus there exist $x \in R^m$ and $\alpha \in R$ such that

$$ux^T > \alpha \geqq vx^T, \qquad \text{all } v \in V.$$

Now, if $\pi \in R^n_+$ then $v = \pi A^T$ is in V (taking $x = 0$ in the definition of V). Thus $xA\pi^T = \pi A^T x^T \leqq \alpha$ for all $\pi \in R^n_+$, and $xA \leqq 0$, $x \in K$. Also $v = 0$ is in V, so that $\alpha \geqq 0$. If $u \in U$ then $0 \leqq \alpha < ux^T \leqq (xCx^T)^{1/2}$, thus $xCx^T > 0$ and

$$v = \frac{xC}{(xCx^T)^{1/2}} \in V,$$

consequently,

$$(xCx^T)^{1/2} > \alpha \geqq \frac{xCx^T}{(xCx^T)^{1/2}} = (xCx^T)^{1/2}$$

a contradiction. Thus $u \notin U$. q.e.d.

Note. A direct application of Lemmas 2 and 5 yields the theorem stated at the beginning.

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