CONCENTRIC TORI IN THE 3-SPHERE

BY C. H. EDWARDS, JR.

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It is proved in this paper that, if A, B, and C are tame solid tori in the 3-sphere S^3 with $A \subset \text{Int } B$ and $B \subset \text{Int } C$, then A and C are concentric if and only if B is concentric with both A and C. It follows from this result that S^3 does not contain an uncountable collection of mutually disjoint tori, no two of which are concentric.

A *torus* is the topological product of two circles, while a *solid torus* is the topological product of a circle and a disk. Two solid tori B and B^*, with $B \subset \text{Int } B^*$, are said to be *concentric* if $\text{Cl}(B^* - B)$ is the topological product of a torus and an interval, while two tori T and T^* in S^3 are *concentric* if they are the boundaries of two concentric solid tori B and B^* respectively in S^3. By a *meridianal disk* of the polyhedral solid torus B is meant a polyhedral disk D, with $\text{Int } D \subset \text{Int } B$ and $\text{Bd } D \subset \text{Bd } B$, such that $\text{Bd } D$ is non-nullhomologous on $\text{Bd } B$. Now let D and E be disjoint meridianal disks of the polyhedral solid torus B, and let K_1 and K_2 be the closures of the two components of $B - (D \cup E)$. Suppose that u_1 and u_2 are unknotted polygonal chords of the 3-cells K_1 and K_2 respectively, each with endpoints $x \in \text{Int } D$ and $y \in \text{Int } E$. Then the simple closed polygon $u_1 \cup u_2$ is called a *center line* of B.

If k is a simple closed polygon interior to the polyhedral solid torus B in S^3, then the *order of B with respect to k*, denoted by $O(B, k)$, is defined to be the minimal number of points of $k \cap D$, for all meridianal disks D of B [5]. If B and B^* are two polyhedral solid tori in S^3, with $B \subset \text{Int } B^*$, then the *order of B^* with respect to B*, denoted by $O(B^*, B)$, is defined to be the order of B^* with respect to an arbitrary center line of B [5]. The two polyhedral solid tori B and B^* in S^3 are said to be *equivalently knotted* if and only if any two center lines c of B and c^* of B^* can be so oriented as to represent the same knot (the same equivalence class of oriented closed polygons under orientation-preserving semilinear autohomeomorphisms of S^3) [5].

A characterization of the relation of concentricity is provided by

LEMMA 1. *Suppose that B and B^* are two polyhedral solid tori in S^3 with $B \subset \text{Int } B^*$. Then B and B^* are concentric if and only if they are equivalently knotted with $O(B^*, B) = 1$.*

Lemma 1 is proved by using some results of Schubert [5] on polyhedral solid tori to sharpen the concentric toral theorem of Harrold, Griffith, and Posey [4].

THEOREM 1. *Suppose that A, B, and C are tame solid tori in S^3 with $A \subset \operatorname{Int} B$ and $B \subset \operatorname{Int} C$. Then A and C are concentric if and only if B is concentric with both A and C.*

PROOF. Since there is a homeomorphism of S^3 onto itself carrying A, B, and C onto polyhedral solid tori, it may be assumed that A, B, and C are polyhedral.

Suppose first that A and C are concentric. It follows from Lemma 1 that A and C are equivalently knotted with $O(C, A)=1$. A theorem of Schubert [5, p. 175] then implies that $O(C, B)O(B, A)=O(C, A)=1$, so that $O(C, B)=O(B, A)=1$, since the order of one solid torus with respect to another is a non-negative integer.

It remains to be shown that A, B, and C are equivalently knotted. Since A and C are concentric, they have a common center line a. Let b be a center line of B, with a and b so oriented that they are homologous in B. Denote by $\bar{a}$ and $\bar{b}$ the knots in S^3 represented by a and b respectively. Since $a \subset \operatorname{Int} B$ with $O(B, a)=1$, a second theorem of Schubert [5, p. 171] implies that $\bar{a}=\bar{b}\bar{x}$ for some knot $\bar{x}$. Since $b \subset \operatorname{Int} C$ with $O(C, b)=1$, the same theorem implies that $\bar{b}=\bar{a}\bar{y}$ for some knot $\bar{y}$. The knot product used here is that defined by Schubert [5] as follows: Suppose that $\bar{k}_1$ and $\bar{k}_2$ are any two knots in S^3. Let S be a polyhedral 2-sphere in S^3 with complementary domains D_1 and D_2, and let w be a polygonal arc on S with endpoints p and q. Let u_1 and u_2 be oriented chords of D_1 and D_2 respectively, both with endpoints p and q, u_1 directed from p to q and u_2 from q to p, such that $u_i \cup w$ (oriented coherently with u_i) represents the knot $\bar{k}_i$, $i=1, 2$. The knot represented by the oriented polygon $u_1 \cup u_2$ is then defined to be the product $\bar{k}_1\bar{k}_2$ of the knots $\bar{k}_1$ and $\bar{k}_2$.

The properties of this product are such that the relations $\bar{a}=\bar{b}\bar{x}$ and $\bar{b}=\bar{a}\bar{y}$ imply that $\bar{a}=\bar{b}$, so that the solid tori A, B, and C are equivalently knotted. It now follows from Lemma 1 that B is concentric with both A and C.

To prove the converse, suppose that B is concentric with both A and C. Then, by Lemma 1, A and C are equivalently knotted with $O(C, A)=O(C, B)O(B, A)=1$, so that A and C are concentric.

COROLLARY 1. *Suppose that $\{B_n\}_1^\infty$ is a sequence of tame solid tori in S^3, with $B_{n+1} \subset \operatorname{Int} B_n$ for $n \geqq 1$, such that $A=\bigcap_{n=1}^\infty B_n$ is a tame solid torus. Then there exists an integer N such that A and B_n are concentric for $n \geqq N$.*

PROOF. Choose a tame solid torus C such that $A \subset \operatorname{Int} C$ with A and C concentric. If N is a positive integer sufficiently large that $B_n \subset \operatorname{Int} C$ for $n \geqq N$, then Theorem 1 implies that A and B_n are concentric for $n \geqq N$.

THEOREM 2. *The 3-sphere S^3 does not contain an uncountable collection of mutually disjoint tori, no two of which are concentric.*

PROOF. Suppose that G is an uncountable collection of mutually disjoint tori in S^3. Since Bing [3] has shown that S^3 does not contain uncountably many mutually disjoint wild closed surfaces, it may be assumed that each torus in G is tame. Therefore, by a theorem of Alexander [1], there may be assigned to each torus $T_\alpha \in G$ a solid torus B_α in S^3 such that $T_\alpha = \text{Bd } B_\alpha$. It may also be assumed without loss that, given T_α and T_β in G, either $B_\alpha \subset \text{Int } B_\beta$ or $B_\beta \subset \text{Int } B_\alpha$, so that G can be linearly ordered by defining $T_\alpha < T_\beta$ if and only if $B_\alpha \subset \text{Int } B_\beta$.

By a theorem of Whyburn [6], G contains an uncountable subcollection G^* such that, to each torus $T \in G^*$ and each point $p \in S^3 - T$, there corresponds a torus $T' \in G^*$ which separates T and p in S^3. Hence let T_0 be an element of this uncountable subcollection G^*, and denote by B_0 the corresponding solid torus bounded by T_0 (as assigned above). Now let C be a tame solid torus concentric with B_0 and containing B_0 in its interior. Then, given any point $p \in \text{Bd } C$, there is a torus T_p in G^* separating p and T in S^3. It follows by an elementary compactness argument, using the linear order introduced above, that there is a torus T_γ in G^* such that $B_0 \subset \text{Int } B_\gamma$ and $B_\gamma \subset \text{Int } C$, if B_γ is the assigned solid torus bounded by T_γ.

Theorem 1 now applies to show that B_0 and B_γ are concentric, so that G contains pairs of concentric tori. With G assumed to be linearly ordered as indicated above, Theorem 1 implies that the relation of concentricity is transitive in G, so that a countability argument may be used to show that G contains an uncountable subcollection G', such that any two tori in G' are concentric.

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