AN APPROXIMATION THEOREM FOR SEMI-GROUPS OF OPERATORS

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Let X be a real or complex Banach space with elements having norm $|f|$, let $E(X)$ be the algebra of bounded linear transformations of X into itself and $\{T(t)\}$ a one-parameter semi-group in $E(X)$ of class $(1, C_1)$:

(i) $T(t)\in E(X)$ for $t\in[0, \infty)$, $T(0)=I$ (identity).

(ii) $T(s+t)=T(s)T(t)$ for $s, t\in[0, \infty)$.

(iii) $\lim_{t\to+0}\left|(1/t)\int_0^t T(\tau)f d\tau - f\right| = 0$ for all $f\in X$.

(iv) $\int_0^1 \|T(\tau)\| d\tau < \infty$, where $\|T\|$ denotes the norm of the operator T.

The infinitesimal operator of a semi-group $\{T(t)\}$ is the linear transformation A defined by

$$Af = \lim_{t\to+0} \frac{T(t) - I}{t} f$$

for all f, for which the limit exists (in the norm topology). It is easy to verify that for all $f\in D(A^p)$, where $D(A^p)$ is the domain of the iterated operator $A^p = A\cdot A^{p-1}$,

$$\frac{d^p}{dt^p} T(t)f = T(t)A^p f = A^p T(t)f, \qquad t \geqq 0.$$

If $f\in D(A^p)$ we have the generalized Taylor's formula

$$T(t)f - \sum_{k=0}^{p-1} \frac{t^k}{k!} A^k f = \frac{1}{(p-1)!}\int_0^t (t-\tau)^{p-1} T(\tau) A^p f d\tau.$$

It is our object to approximate $T(t)f$ for $f\in D(A^{p-1})$ by the Taylor-polynomial $\sum_{k=0}^{p-1} (t^k/k!)A^k f$, giving:

THEOREM. *Let $\{T(t)\}$ be a semi-group of class $(1, C_1)$, $f_0\in D(A^{p-1})$.*
(a) *If*

$$\liminf_{t\to+0}\left|\frac{p!}{t^p}\left(T(t) - \sum_{k=0}^{p-1}\frac{t^k}{k!}A^k\right)f_0 - g_0\right| = 0,$$

then $f_0\in D(A^p)$ and $A^p f_0 = g_0$. If

$$\liminf_{t\to+0}\left|\frac{p!}{t^p}\left(T(t) - \sum_{k=0}^{p-1}\frac{t^k}{k!}A^k\right)f_0\right| = 0,$$

then $T(t)f_0 \equiv \sum_{k=0}^{p-1} (t^k/k!)A^k f_0$.

(b)
$$T(t)f_0 - \sum_{k=0}^{p-1} \frac{t^k}{k!} A^k f_0 = o(t^{p-1}).$$

(c) *If X is reflexive and*

$$\liminf_{t\to+0} \left| \frac{p!}{t^p}\left(T(t) - \sum_{k=0}^{p-1} \frac{t^k}{k!} A^k \right) f_0 \right|$$

is finite, then $f_0 \in D(A^p)$ and

$$A^p f_0 = \lim_{t\to+0} \frac{p!}{t^p}\left(T(t) - \sum_{k=0}^{p-1} \frac{t^k}{k!} A^k \right) f_0.$$

In case $p=1$, resp. $p=2$ in part (b), the above theorem is essentially identical with a result contained in Hille and Phillips [**3**, p. 326], parts (a) for $p=1$ and (b) for $p=2$ being due to Hille [**2**, p. 323] and part (c) for $p=1$ to Butzer [**1**]. We remark that de Leeuw [**4**] has considered the approximation by adjoint semi-groups so as to give results for part (c) (with $p=1$) in the case of non-reflexive Banach spaces.

Sketch of proof. Fundamental is the identity

$$(*) \qquad \frac{1}{s}\int_0^s T(\tau) B_t^p f d\tau = \frac{T(s) - I}{s} \int_0^t p \frac{\tau^{p-1}}{t^p} B_\tau^{p-1} f d\tau$$

where

$$B_t^p f = \frac{p!}{t^p}\left(T(t) - \sum_{k=0}^{p-1} \frac{t^k}{k!} A^k \right) f.$$

In the limit $s \to +0$ we get

$$B_t^p f = \int_0^t p \frac{\tau^{p-1}}{t^p} B_\tau^{p-1} A f d\tau,$$

which gives by induction on p that $B_t^p f = A^p f + o(1)$ for $f \in D(A^p)$ which is equivalent to (b). Under the hypothesis of (a) we obtain from (*)

$$\left| \frac{T(s) - I}{s} A^{p-1} f_0 - \frac{1}{s}\int_0^s T(\tau) g_0 d\tau \right|$$
$$\leqq \liminf_{t\to+0} \left| \frac{1}{s}\int_0^s T(\tau)(B_t^p f_0 - g_0) d\tau \right| = 0,$$

giving the result (a).

Under the hypothesis of (c) there exists a sequence $t_\nu \to +0$ such that $B^p_{t_\nu} f_0$ is weakly convergent to an element g_0. Then also $(1/s)\int_0^s T(\tau) B^p_{t_\nu} f_0 d\tau$ converges weakly to g_0 and the relation (*) can be used to show that

$$\frac{T(s) - I}{s} A^{p-1} f_0 = \frac{1}{s} \int_0^s T(\tau) g_0 d\tau \to g_0 = A^p f_0.$$

These results have various applications to the solutions of partial differential equations, e.g. the heat equation. The proofs of these and further results will appear elsewhere.

References

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2. E. Hille, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloquium Publications, vol. 31, New York, 1948.

3. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, rev. ed., New York, 1957.

4. K. de Leeuw, *On the adjoint semi-group and some problems in the theory of approximation*, Math. Z. vol. 73 (1960).

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