GENERALIZATION OF CERTAIN THEOREMS OF G. SZEGÖ ON THE LOCATION OF ZEROS OF POLYNOMIALS

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Using the theorem of Grace, the following result was obtained by G. Szegö:[1]

Let the polynomial

$$f(z) = z^n + A_1 z^{n-1} + \cdots + A_n$$

have no zeros in the circular region $|z| \leqq R$. *Then the "section"*

$$h(z) = f(z) - z^n = A_1 z^{n-1} + A_2 z^{n-2} + \cdots + A_n$$

has no zeros in the circular region $|z| \leqq R/2$.

In case of an even n the example $f(z) = (z-R)^n$ shows that the circle $|z| \leqq R/2$ can not be replaced by a larger concentric circle. But in case n is odd, according to Szegö the polynomial $h(z)$ is different from zero even in the circle $|z| \leqq (R/2) \sec (\pi/2n)$.[2]

This theorem can be generalized as follows:

I. *Let the polynomial*

$$f(z) = z^n + A_1 z^{n-1} + \cdots + A_n$$

have no zeros in the circular region $|z-\alpha| \leqq R$. *Then no polynomial*

$$h(z) = f(z) - \epsilon(z - \alpha)^n, \qquad |\epsilon| \leqq 1,$$

can have any zeros in the circle $|z-\alpha| \leqq R/2$.

The example $f(z) = (z-R)^n$, $\alpha = 0$, $\epsilon = 1$, shows that this theorem can not be refined even in the case of an odd n.

Theorem I is a consequence of the following more general theorem:

II. *Let the polynomial*

$$f(z) = (z - a_1)(z - a_2) \cdots (z - a_n) \tag{1}$$

have no zeros in the circle $|z-\alpha| \leqq R$; *and let the polynomial*

$$g(z) = (z - b_1)(z - b_2) \cdots (z - b_n) \tag{2}$$

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[1] G. Szegö, *Bemerkungen zu einem Satz von J. H. Grace über die Wurzeln algebraischer Gleichungen*, Math. Zeit. vol. 13 (1922) pp. 28–55.

[2] Loc. cit. p. 46.

have all its zeros in the circle $|z-\alpha| \leqq \rho$, $\rho < R$. *Then the polynomial*

$$h(z) = f(z) - \lambda g(z), \qquad |\lambda| \leqq t^n,\ 0 \leqq t < R/\rho, \tag{3}$$

can have no zero in the circle

$$|z - \alpha| \leqq r = \frac{R - \rho t}{1 + t} \cdot \tag{4}$$

In order to prove this we observe first that for any zero ξ of the polynomial $h(z)$

$$\frac{f(\xi)}{g(\xi)} = \prod_{k=1}^{n} \frac{\xi - a_k}{\xi - b_k} = \lambda.$$

Hence $h(z_0) \neq 0$ in every point z_0 where

$$\left|\frac{f(z_0)}{g(z_0)}\right| = \prod_{k=1}^{n}\left|\frac{z_0 - a_k}{z_0 - b_k}\right| \neq |\lambda|.$$

Now at every point z_0 of the circular region (4)

$$\begin{aligned} |z_0 - a_k| &\geqq |a_k - \alpha| - |z_0 - \alpha| \geqq |a_k - \alpha| - r > R - r \\ &= (r + \rho)t, \\ |z_0 - b_k| &\leqq |z_0 - \alpha| + |b_k - \alpha| \leqq r + \rho, \end{aligned}$$

so that

$$\left|\frac{f(z_0)}{g(z_0)}\right| = \prod_{k=1}^{n}\left|\frac{z_0 - a_k}{z_0 - b_k}\right| > \left(\frac{R - r}{r + \rho}\right)^n = t^n \geqq |\lambda|.$$

This concludes the proof of II. In the case $b_1 = b_2 = \cdots = b_n = \alpha$, $|\lambda| = |\epsilon| = 1$ $(\rho = 0,\ t = 1)$, we obtain I.

Substituting

$$g(z) = z^n + A_k z^{n-k} = z^{n-k}(z^k + A_k), \quad \alpha = 0, \quad \rho = |A_k|^{1/k}, \quad \lambda = 1$$

in Theorem II we obtain:

III. *The polynomial*

$$f(z) = z^n + A_k z^{n-k} + A_{k+1} z^{n-k-1} + \cdots + A_n$$

has at least one zero in the circle $|z| \leqq 2r + |A_k|^{1/k}$ *provided the section*

$$h(z) = f(z) - z^n - A_k z^{n-k} \equiv A_{k+1} z^{n-k-1} + A_{k+2} z^{n-k-2} + \cdots + A_n$$

has at least one zero on the circle $|z| \leqq r$.

The proof of the following theorem is similar to that of II:

IV. *Let $a_1, a_2, \cdots, a_n$ be given points of the complex plane which are all different from α. Let S_k be the half-plane containing α and having as boundary the perpendicular bisector of $\alpha_1\, a_k$. Finally let S^* be the common part of the half-planes: $S_1, S_2, \cdots, S_n$. Then no polynomial*

$$h(z) \equiv (z - a_1)(z - a_2) \cdots (z - a_n) - \epsilon(z - \alpha)^n, \qquad |\epsilon| \leqq 1,$$

can have a zero in the interior of the convex domain S^*.

Indeed in any point z_0 in the interior of S^* we have $|z_0 - a_k| > |z_0 - \alpha|$ so that

$$\left|\frac{f(z_0)}{g(z_0)}\right| = \prod_{k=1}^{n}\left|\frac{z_0 - a_k}{z_0 - \alpha_k}\right| > 1 \geqq |\epsilon|.$$

A theorem similar to II holds also if the zeros of the polynomials $f(z)$ and $g(z)$ are in arbitrary circular domains without common points. One of these circular domains is the interior of a circle, the other the exterior or interior of a circle or a half-plane. Corresponding to these cases three theorems can be obtained generalizing also certain theorems of G. Szegö.[3]

V. *Let the zeros of the polynomials $f(z) = (z - a_1) \cdots (z - a_n)$ and $g(z) = (z - b_1)(z - b_2) \cdots (z - b_n)$ be located in the circular regions*

$$|z - \alpha| \geqq \rho_1 \quad \textit{and} \quad |z - \beta| \leqq \rho_2, \tag{5}$$

respectively. We assume that these regions have no points in common, that is,

$$\rho_1 - \rho_2 > 0, \qquad |\beta - \alpha| < \rho_1 - \rho_2. \tag{6}$$

Then no polynomial

$$h(z) = f(z) - \epsilon g(z), \qquad |\epsilon| \leqq 1, \tag{7}$$

can have a zero in the interior of the ellipse E with foci at α and β and with the major axis $\rho_1 - \rho_2$.

VI. *Let the zeros of the polynomials $f(z)$ and $g(z)$ be located in the circular regions*

$$|z - \alpha| \leqq \rho_1 \quad \textit{and} \quad |z - \beta| \leqq \rho_2, \tag{8}$$

respectively, such that these regions have no points in common, that is,

$$|\beta - \alpha| > \rho_1 + \rho_2. \tag{9}$$

Then no polynomial

[3] Loc. cit. pp. 47–48, Theorems 13–15.

$$h(z) = f(z) - \epsilon g(z), \qquad |\epsilon| = 1,$$

can have a zero in the interior of the hyperbola H with foci at α and β and with the real axis $\rho_1+\rho_2$.

VII. *Let the zeros of the polynomials $f(z)$ and $g(z)$ be located in the circular region $|z-\alpha| \leqq \rho$ and in the half-plane S, respectively, such that these regions have no points in common. Let K be a conic section with α as focus and the boundary line L of the half-plane S as the directrix corresponding to α.*[4] *Then no polynomial*

$$f(z) - \lambda g(z) \tag{10}$$

with

$$|\lambda| \geqq t^n = \left(e + \rho\frac{e+1}{\delta}\right)^n \tag{11}$$

can have a zero in the interior of the conic section K where e is the numerical eccentricity of K and δ is the distance of α from the line L.

By the interior of a conic section we mean the set of points from which no tangent can be drawn to the given conic section.

In order to prove V and VI, we denote by z_0 an arbitrary point in the interior of the conic sections E and H, respectively; let

$$|z_0 - \alpha| = r_1, \qquad |z_0 - \beta| = r_2.$$

As to the ellipse E we have

$$r_1 + r_2 < \rho_1 - \rho_2 \quad \text{or} \quad \rho_2 + r_2 < \rho_1 - r_1,$$
$$|z_0 - a_k| \geqq |a_k - \alpha| - |z_0 - \alpha| = |a_k - \alpha| - r_1 > \rho_1 - r_1,$$
$$|z_0 - b_k| \leqq |b_k - \beta| + |z_0 - \beta| \leqq \rho_2 + r_2.$$

From this Theorem V follows since

$$\left|\frac{z_0 - a_k}{z_0 - b_k}\right| > \frac{\rho_1 - r_1}{\rho_2 + r_2} > 1 \quad \text{so that} \quad \left|\frac{f(z_0)}{g(z_0)}\right| > 1 \geqq |\epsilon|.$$

As to the hyperbola H, we distinguish two cases regarding the position of z_0, according as z_0 is nearer to β than to α, or conversely. In the first case

$$r_1 > r_2, \quad r_1 - r_2 > \rho_1 + \rho_2, \quad \text{hence} \quad r_1 - \rho_1 > r_2 + \rho_2,$$
$$|z_0 - a_k| \geqq |z_0 - \alpha| - |a_k - \alpha| \geqq r_1 - \rho_1,$$
$$|z_0 - b_k| \leqq |z_0 - \beta| + |b_k - \beta| \leqq r_2 + \rho_2.$$

[4] That is, the polar of α.

In the second case

$$r_1 < r_2, \quad r_2 - r_1 > \rho_1 + \rho_2, \quad \text{hence} \quad r_2 - \rho_2 > r_1 + \rho_1,$$
$$|z_0 - a_k| \leqq |z_0 - \alpha| + |a_k - \alpha| \leqq r_1 + \rho_1,$$
$$|z_0 - b_k| \geqq |z_0 - \beta| - |b_k - \beta| \geqq r_2 + \rho_2.$$

Thus we have in the first and second case,

$$\left|\frac{z_0 - a_k}{z_0 - b_k}\right| \geqq \frac{r_1 - \rho_1}{r_2 + \rho_2} > 1, \quad \text{and} \quad \left|\frac{z_0 - a_k}{z_0 - b_k}\right| \leqq \frac{r_1 + \rho_1}{r_2 - \rho_2} < 1,$$

respectively. This furnishes Theorem VI since we have

$$\left|\frac{f(z_0)}{g(z_0)}\right| > 1 = |\epsilon|, \quad \text{and} \quad \left|\frac{f(z_0)}{g(z_0)}\right| < 1 = |\epsilon|,$$

respectively.

This proof furnishes the following corollary:

VI′. *Let H be the hyperbola defined in Theorem* VI. *No polynomial $f(z)-\lambda g(z)$ with $|\lambda| \geqq 1$ ($|\lambda| \leqq 1$) can have any zero inside the branch of H containing the focus α (β).*

Let z_0 be a point in the interior of the conic section K defined in Theorem VII; let r and d be the distance of z_0 from the focus α and the directrix L, respectively. Then

$$\frac{r}{d} < e \quad \text{and} \quad \frac{1}{d} < \frac{1}{d^*} = \frac{e+1}{\delta};$$

indeed if z^* is the point of K nearest to L and d^* is the distance of z^* from L, we have

$$\frac{|z^* - \alpha|}{d^*} = \frac{\delta - d^*}{d^*} = e.$$

But $|z_0-a_k| \leqq |z_0-\alpha| + |a_k-\alpha| \leqq r+\rho$ and $|z_0-b_k| \geqq d$, so that

$$\left|\frac{z_0 - a_k}{z_0 - b_k}\right| \leqq \frac{r+\rho}{d} < e + \frac{\rho}{d^*} = e + \rho\frac{e+1}{\delta} = t,$$
$$\left|\frac{f(z_0)}{g(z_0)}\right| \leqq \left(\frac{r+\rho}{d}\right)^n < t^n = |\lambda|.$$

This establishes the proof of Theorem VII.

The special cases

$$b_1 = b_2 = \cdots = b_n = 0, \beta = 0, \rho_2 = 0, \epsilon = 1 \qquad \text{of V and VI,}$$

and

$$a_1 = a_2 = \cdots = a_n = 0, \alpha = 0, \rho = 0, e = 1, \lambda = 1 \qquad \text{of VII,}$$

are due to G. Szegö.

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SOME REMARKS ON POLYNOMIALS

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This note contains some disconnected remarks on polynomials.

Let $f_n(x)=\prod_{i=1}^n(x-x_i)$, $-1\leqq x_1\leqq x_2\leqq \cdots \leqq x_n\leqq 1$. Denote by $-1\leqq y_1\leqq \cdots \leqq y_{n-1}\leqq 1$ the roots of $f_n'(x)$. We prove the following theorem.

THEOREM 1. *For all n*

$$(1) \qquad |f_n(-1)| + |f_n(+1)| + \sum_{i=1}^{n-1} |f_n(y_i)| \leqq 2^n.$$

For $n\geqq 3$

$$(2) \qquad |f_n(-1)|^{1/2} + |f_n(+1)|^{1/2} + \sum_{i=1}^{n-1} |f_n(y_i)|^{1/2} \leqq 2^{n/2}.$$

For $n\geqq n_0(k)$

$$(3) \qquad |f_n(-1)|^{1/k} + |f_n(+1)|^{1/k} + \sum_{i=1}^{n-1} |f_n(y_i)|^{1/k} \leqq 2^{n/k}.$$

REMARK. If $y_i=y_{i+1}$ or $-1=y_1$, $+1=y_{n-1}$ the corresponding summands clearly vanish.

Clearly

$$|f_n(-1)| \leqq (1-x_1)2^{n-1}, \qquad |f_n(y_i)| \leqq |y_i - x_{i+1}| 2^{n-1},$$
$$|f_n(+1)| \leqq (1-x_n)2^{n-1}.$$

Thus

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